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MULTIPLICATIVE TRANSITION SYSTEMS

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Abstract.

The paper is concerned with algebras whose elements can be used to represent runs of a system from a state to a state. These algebras, called multiplicative transition systems, are categories with respect to a partial binary operation called composition. They can be characterized by axioms such that their elements and operations can be represented by partially ordered multisets of a certain type and operations on such multisets. The representation can be obtained without assuming a discrete nature of represented elements. In particular, it remains valid for systems with infinitely divisible elements, and thus also for systems with elements which can represent continuous and partially continuous runs.

Key words

Transition systems, states, transitions, composition, category, independence, regions, labelled posets, pomsets.

MULTIPLIKATYWNE SYSTEMY TRANZYCYJNE

Streszczenie.

Praca dotyczy algebr, których elementy mogą być użyte do reprezentowania przebiegów dowolnego systemu od stanu do stanu. Te algebry, zwane multiplikatywnymi systemami tranzycyjnymi, są kategoriami ze względu na częściową operację binarną zwaną składaniem. Można je scharakteryzować aksjomatami tak, że ich elementy i operację składania można reprezentować częściowo uporządkowanymi wielozbiorami pewnego typu i operację składania takich wielozbiorów. Taką reprezentację można otrzymać bez zakładania dyskretności reprezentowanych elementów. W szczególności jest ona możliwa dla systemów o nieskończoności podzielnych elementach, a więc i dla systemów, których elementy mogą reprezentować przebiegi ciągłe i częściowo ciągłe.

Słowa kluczowe

Systemy tranzycyjne, stany, tranzycje, składanie, kategoria, niezależność, regiony, etykietowane zbiorzy częściowo uporządkowane, wieloziobory częściowo uporządkowane.
1 Introduction

This paper is an attempt to develop a mathematical framework for describing systems that may exhibit arbitrary combination of discrete and continuous behaviour.

There are reasons for which we need such a universal framework.

First, in order to describe and analyse systems including computer components, which operate in discrete steps, and real-world components, which operate in a continuous way, we need a framework including ideas from both computer science and control theory (cf. [LSV 07]). Consequently, we need a simple language to describe in the same way and to relate behaviours of systems of any nature, including discrete, continuous, and hybrid systems. Second, we need basic axioms valid for systems of any nature such that every particular subclass of systems could be characterized by only adding to the list of basic axioms the respective specific axioms. Third, we need a representation theorem resulting in a representation of system runs by well defined mathematical structures and in a representation of the composition of system runs by a composition of such structures. In particular, we need runs of discrete, continuous, and hybrid systems to be represented by structures of the same type. This will allow us to avoid inventing a special representation in every particular case.

Our idea of a universal framework for describing systems consists in a generalization of the concept of a transition system.

Transition systems are models of systems which operate in discrete steps (cf. [RT 86] and [NRT 90]). They specify system states and transitions between states, the latter supposed to be indivisible. Consequently, they have means to represent implicitly partial and complete system runs viewed as sequences of successive transitions. They can be provided in a natural way with a composition of runs of which one starts from the final state of the other, and this results in the structure of a category.

In the case of systems with continuous behaviour runs cannot be viewed as sequences of discrete steps. Nevertheless, the concept of a run still makes sense, and there is a natural composition of runs of which one starts from the resulting state of the other. Moreover, the continuity can be expressed as infinite divisibility of runs with respect to such a composition.

Thus the category consisting of system states and runs, and of the respective composition, is a good candidate for a universal structure allowing one to represent both discrete and continuous behaviour. We call it a multiplicative transition system and call system runs represented in it generalized transitions. Note that the concept of a multiplicative transition system gener-
alizes the standard concept of a transition system in the sense that every usual transition system can be regarded as the set of generators of the multiplicative transition system of the respective runs.

In order to formulate basic axioms characterizing multiplicative transition systems we refer to the concepts and properties of processes and operations on processes described in [Wink 09a].

2 Processes

In order to represent states and runs of systems in [Wink 09a] a concept of a process in a universe of objects has been introduced, some operations on processes in a universe of objects have been defined, and the respective algebras of processes have been defined and studied.

A universe of objects has been defined as \( U = (W, V, ob) \), where \( V \) is a set of objects, \( W \) is a set of instances of objects from \( V \) (a set of object instances), and \( ob \) is a mapping that assigns the respective object to each of its instances.

A concrete process in \( U \) has been defined as a labelled partially ordered set \( L = (X, \leq, \text{ins}) \), where \( X \) is a set (of occurrences of objects from \( V \), called object occurrences), \( \text{ins} : X \to W \) is a mapping (a labelling that assigns an object instance to each occurrence of the respective object), \( \leq \) is a partial order (the flow order of \( L \)) such that: (1) for every object \( v \in V \), the set \( X|v = \{ x \in X : ob(\text{ins}(x)) = v \} \) is either empty or it is a maximal chain and has an element in every cross-section, (2) every element of \( X \) belongs to a cross-section, (3) no segment of \( L \) is isomorphic to its proper subsegment, and where a cross-section has been defined as a maximal antichain \( Z \) such that the reflexive and transitive closure of the union of the restrictions of the partial order \( \leq \) to \( X^- (Z) = \{ x \in X : x \leq z \text{ for some } z \in Z \} \) and to \( X^+ (Z) = \{ x \in X : z \leq x \text{ for some } z \in Z \} \) is exactly the partial order \( \leq \).

An abstract process has been defined as an isomorphism class \( \xi \) of concrete processes. Each member \( L \) of such a class has been called an instance of this class and \( \xi \) has been written as \([L]\).

The set of objects occurring in a cross-section is the same for all cross-sections of \( L \). It has been called the range of \( L \) and written as \( \text{objects}(L) \). It has been said that \( L \) is global if \( \text{objects}(L) = V \). It has been said that \( L \) is bounded if the set of elements of \( L \) that are minimal with respect to \( \leq \) and the set of elements of \( L \) that are maximal with respect to \( \leq \) are cross-sections; the respective cross-sections have been called the origin and the end of \( L \), and they have been written as \( \text{origin}(L) \) and \( \text{end}(L) \).
For every concrete process \( L' \) such that \( L \) and \( L' \) are isomorphic the sets \( \text{objects}(L') \) and \( \text{objects}(L) \) are the same. Consequently, for the abstract process \( [L] \) that corresponds to a concrete process \( L \), one can define \( \text{objects}([L]) = \text{objects}(L) \). An abstract process is said to be global (resp.: bounded) if the instances of this process are global (resp.: bounded).

2.1. Example. Consider a producer \( p \) that produces some material for a distributor \( d \). Define an instance of \( p \) to be a pair \((p,q)\), where \( q \geq 0 \) is the amount of material at disposal of \( p \). Define an instance of \( d \) to be a pair \((d,r)\), where \( r \geq 0 \) is the amount of material at disposal of \( d \). Define \( V' = \{p,d\} \), \( W' = W'_p \cup W'_d \), where \( W'_p = \{(p,q): q \geq 0\} \), \( W'_d = \{(d,r): r \geq 0\} \). Define \( ob'(w) = p \) for \( w = (p,q) \in W'_p \) and \( ob'(w) = d \) for \( w = (d,r) \in W'_d \). Then \( U' = (W', V', ob') \) is a universe of objects.

Undisturbed production of material by the producer \( p \) in an interval \([t', t'']\) of global time is a concrete process in \( U' \) that can be defined as

\[
Q = (X_Q, \leq_Q, \text{ins}_Q) \text{ where}
\]

\( X_Q \) is the set of numbers equal to variations \( \text{var}(t \mapsto q(t); t', t) \) in \([t', t] \subseteq [t', t'']\) of the real valued function \( t \mapsto q(t) \) that specifies the amount of material at disposal of \( p \) at every moment of \([t', t'']\), \( \leq_Q \) is the restriction of the usual order of numbers to \( X_Q \), \( \text{ins}_Q(x) = (p, q(t)) \) for \( x = \text{var}(t \mapsto q(t); t', t) \).

(We remind that the variation of a real-valued function \( f \) on an interval \([a, b]\), written as \( \text{var}(f; a, b) \), is the least upper bound of the set of numbers \( |f(a_1) - f(a_0)| + \ldots + |f(a_n) - f(a_{n-1})| \) corresponding to subdivisions \( a = a_0 < a_1 < \ldots < a_n = b \) of \([a, b]\). In the case of more than one real-valued function the concept of variation turns into the concept of the length of the curve defined by these functions.).

Defining \( X_Q \) as above instead of taking simply \( X_Q \) equal to \([t', t'']\) is necessary in order to ensure the condition (3) of process definition (this condition could not be ensured with \( X_Q = [t', t''] \) if the function \( t \mapsto q(t) \) were constant on subsegments of \([t', t'']\)).

Undisturbed distribution of material by the distributor \( d \) in an interval \([t', t'']\) of global time is a concrete process in \( U' \) that can be defined as

\[
R = (X_R, \leq_R, \text{ins}_R) \text{ where}
\]

\( X_R \) is the set of numbers equal to variations \( \text{var}(t \mapsto r(t); t', t) \) in \([t', t] \subseteq [t', t'']\) of the real valued function \( t \mapsto r(t) \) that specifies the amount of material at disposal of \( d \) at every moment of \([t', t'']\),
\( \leq_R \) is the restriction of the usual order of numbers to \( X_R \),
\( \text{ins}_R(x) = (d, r(t)) \) for \( x = \text{var}(t \mapsto r(t); t', t) \).

Transfer of an amount \( m \) of material from the producer \( p \) to the distributor \( d \) is a concrete process in \( U' \) that can be defined as \( S = (X_S, \leq_S, \text{ins}_S) \), where \( X_S = \{x_1, x_2, x_3, x_4\} \),
\[ x_1 <_S x_3, \quad x_2 <_S x_4, \quad x_2 <_S x_3, \quad x_1 <_S x_4, \]
\[ \text{ins}_S(x_1) = (d, r), \quad \text{ins}_S(x_2) = (p, q), \quad \text{ins}_S(x_3) = (d, r + m), \]
\[ \text{ins}_S(x_4) = (p, q - m). \]

Transfer of an amount of material from the producer \( p \) to the distributor \( d \) followed by independent behaviour of \( p \) and \( d \) and by another transfer of material from \( p \) to \( d \) is a concrete process \( T = (X_T, \leq_T, \text{ins}_T) \) in \( U' \), where \( X_T = X_Q' \cup X_R' \cup X_S' \cup X_S'' \),
\[ \leq_T \text{ is the transitive closure of } \leq_Q' \cup \leq_R' \cup \leq_S' \cup \leq_S'', \]
\[ \text{ins}_T = \text{ins}_Q' \cup \text{ins}_R' \cup \text{ins}_S' \cup \text{ins}_S'', \]
for a variant \( Q' \) of \( Q \), a variant \( R' \) of \( R \), and variants \( S' \) and \( S'' \) of \( S \), such that one maximal element of \( X_S' \) coincides with the minimal element of \( X_Q' \) with the same label and the other maximal element coincides with the minimal element of \( X_R' \) with the same label, one minimal element of \( X_S'' \) coincides the maximal element of \( X_Q' \) with the same label and the other minimal element coincides with the maximal element of \( X_R' \) with the same label, and these are the only common elements of pairs of sets from among \( X_Q', X_R', X_S', X_S'' \).

The abstract processes corresponding to the concrete processes \( Q, R, S, \) and \( T \), are represented graphically in figure 2.1. \( \sharp \)
Figure 2.1: \([Q], [R], [S], [T]\)
3 Operations on processes

Collecting concrete processes into isomorphism classes, i.e. making abstract processes, allows one to define some operations on the latter. In what follows, the word “process” means “abstract process”.

Let \( \text{PROC}(U) \) and \( \text{Proc}(U) \) denote respectively the set of all processes in \( U \) and the subset of all bounded processes in \( U \).

In the set \( \text{PROC}(U) \) of processes in \( U \) there exists a bounded process with the empty set of object instances, called the empty process and denoted by 0.

Processes from \( \text{PROC}(U) \) with flow orders reducing to identity relations are bounded, they have been called states, or identities, and they have been identified with the sets of instances of occurring objects.

For each process \( \alpha \) from \( \text{PROC}(U) \) with an instance \( L \in \alpha \) that has the cross-section \( \text{origin}(L) \) (resp.: \( \text{end}(L) \)), there exists the unique identity \( \{\text{origin}(L)\} \) (resp.: \( \{\text{end}(L)\} \)), called the source or the domain of \( \alpha \) and written as \( \text{dom}(\alpha) \) (resp.: called the target or the codomain of \( \alpha \) and written as \( \text{cod}(\alpha) \)).

For each cross-section \( c \) of a concrete process \( L = (X, \leq, \text{ins}) \), the restrictions of \( L \) to the subsets \( X^{-}(c) = \{x \in X : x \leq z \text{ for some } z \in c\} \) and \( X^{+}(c) = \{x \in X : z \leq x \text{ for some } z \in c\} \) are concrete processes, called respectively the head and the tail of \( L \) with respect to \( c \), and written respectively as \( \text{head}(L, c) \) and \( \text{tail}(L, c) \).

A process \( \alpha \) has been said to consist of a process \( \alpha_1 \) followed by a process \( \alpha_2 \), and we have been saying that \( \alpha_1 \) is a full prefix of \( \alpha \), written as \( \alpha_1 \text{ fpref } \alpha \), iff an instance \( L \) of \( \alpha \) has a cross-section \( c \) such that \( \text{head}(L, c) \) is an instance of \( \alpha_1 \) and \( \text{tail}(L, c) \) is an instance of \( \alpha_2 \).

For every two processes \( \alpha_1 \) and \( \alpha_2 \) such that \( \text{cod}(\alpha_1) \) and \( \text{dom}(\alpha_2) \) are defined and \( \text{cod}(\alpha_1) = \text{dom}(\alpha_2) \) there exists a unique process, written as \( \alpha_1;\alpha_2 \), or as \( \alpha_1 \alpha_2 \), that consists of \( \alpha_1 \) followed by \( \alpha_2 \).

The operation \( (\alpha_1, \alpha_2) \mapsto \alpha_1\alpha_2 \) has been called the sequential composition of processes.

A splitting of a concrete process \( L = (X, \leq, \text{ins}) \) has been defined as an ordered pair \( s = (X^F, X^S) \) of two disjoint subsets \( X^F \) and \( X^S \) of \( X \) such that \( X^F \cup X^S = X \), \( x' \leq x'' \) only if \( x' \) and \( x'' \) are both in one of these subsets.

For each splitting \( s = (X^F, X^S) \) of a concrete process \( L = (X, \leq, \text{ins}) \), the restrictions of \( L \) to the subsets \( X^F \) and \( X^S \) are concrete processes, called respectively the first part and the second part of \( L \) with respect to \( s \), and written respectively as \( \text{first}(L, s) \) and \( \text{second}(L, s) \).
A process $\alpha$ has been said to consist of two parallel processes $\alpha_1$ and $\alpha_2$, and we say that $\alpha_1$ and $\alpha_2$ are prefixes of $\alpha$ and of every process $\gamma = \alpha \beta$, written respectively as $\alpha_1 \text{ pref } \alpha$, $\alpha_2 \text{ pref } \alpha$, $\alpha_1 \text{ pref } \gamma$, $\alpha_1 \text{ pref } \gamma$, iff an instance $L$ of $\alpha$ has a splitting $s$ such that $\text{first}(L, s)$ is an instance of $\alpha_1$ and $\text{second}(L, s)$ is an instance of $\alpha_2$.

For every two processes $\alpha_1$ and $\alpha_2$ such that $\text{objects}(\alpha_1) \cap \text{objects}(\alpha_2) = \emptyset$ there exists a process $\alpha$ with an instance $L$ that has a splitting $s$ such that $\text{first}(L, s)$ is an instance of $\alpha_1$ and $\text{second}(L, s)$ is an instance of $\alpha_2$. If such a process $\alpha$ exists then it is unique, it is written it as $\alpha_1 + \alpha_2$, and the processes $\alpha_1$ and $\alpha_2$ are said to be parallel.

The operation $(\alpha_1, \alpha_2) \mapsto \alpha_1 + \alpha_2$ has been called the parallel composition of processes.

The partial algebra $\text{Proc}(U) = (\text{Proc}(U), \text{dom}, \text{cod}, ;, +, 0)$ has been called the algebra of bounded processes in $U$.

3.1. Example. Consider the universe $U'$ of a producer and a distributor and the concrete processes $Q$, $R$, $S$, $Q'$, $R'$, $S'$, $S''$, $T$ in this universe described in example 2.1. Consider the corresponding abstract processes $\pi = [Q']$, $\rho = [R']$, $\sigma' = [S']$, $\sigma'' = [S'']$, $\tau = [T]$. We can represent $\tau$ as $\sigma'(\pi + \rho)\sigma''$. In general, by combining the abstract processes corresponding to the possible variants of concrete processes $Q$, $R$, $S$ we obtain elements of the algebra $\text{Proc}(U')$ of bounded processes in the universe $U'$.

It has been shown that the reduct $\text{cat}((\text{Proc}(U)) = (\text{Proc}(U), \text{dom}, \text{cod}, ;)$ of the algebra $\text{Proc}(U) = (\text{Proc}(U), \text{dom}, \text{cod}, ;, +, 0)$ is an arrows-only category in the sense of [McL 71], and that it enjoys the following properties:

(A1) If $\sigma \alpha$ and $\sigma' \alpha$ are defined and $\sigma \alpha = \sigma' \alpha$ then $\sigma = \sigma'$.

(A2) If $\alpha \tau$ and $\alpha \tau'$ are defined and $\alpha \tau = \alpha \tau'$ then $\tau = \tau'$.

(A3) If $\sigma \tau$ is an identity then $\sigma$ and $\tau$ are also identities.

(A4) If $\sigma \alpha \tau$ is defined and the restriction of the category $\mathcal{A}$ to the set of components of $\sigma \alpha \tau$ is isomorphic to the restriction of $\mathcal{A}$ to the set of components of $\alpha$ then $\sigma$ and $\tau$ are identities.
For all $\xi_1, \xi_2, \eta_1, \eta_2$ such that $\xi_1 \xi_2 = \eta_1 \eta_2$ there exist unique $\sigma_1, \sigma_2$, and a unique bicartesian square $(v_\alpha \overset{\alpha_1}{\leftarrow} u_\alpha \overset{\alpha_2}{\rightarrow} w, v_\alpha' \overset{\alpha_1'}{\leftarrow} u_\alpha' \overset{\alpha_2'}{\rightarrow} w)$ such that $\xi_1 = \sigma_1 \alpha_1, \xi_2 = \sigma'_2 \sigma_2, \eta_1 = \sigma_1 \alpha_2, \eta_2 = \alpha'_2 \sigma_2$.

If $(v_\alpha \overset{\alpha_1}{\leftarrow} u_\alpha \overset{\alpha_2}{\rightarrow} w, v_\alpha' \overset{\alpha_1'}{\leftarrow} u_\alpha' \overset{\alpha_2'}{\rightarrow} w)$ is a bicartesian square then for every decomposition $u_\alpha \overset{\alpha_1}{\rightarrow} v = u_\alpha \overset{\alpha_{1_1}}{\rightarrow} v_1 \overset{\alpha_{1_2}}{\rightarrow} v$ (resp. $w_\alpha \overset{\alpha_1'}{\leftarrow} u_\alpha' = w_\alpha \overset{\alpha'_{1_2}}{\leftarrow} w_{1} \overset{\alpha'_{1_1}}{\rightarrow} u_\alpha'$) there exist a unique decomposition $w_\alpha \overset{\alpha_1'}{\leftarrow} u_\alpha' = w_\alpha \overset{\alpha_{1_1}}{\leftarrow} w_1 \overset{\alpha_{1_2}}{\rightarrow} u_\alpha'$ (resp. $u_\alpha \overset{\alpha_1}{\rightarrow} v = u_\alpha \overset{\alpha_{1_1}}{\rightarrow} v_1 \overset{\alpha_{1_2}}{\rightarrow} v$), and a unique $v_1 \overset{\alpha_{2_2}}{\leftarrow} w_1$, such that $(v_1 \overset{\alpha_{1_1}}{\leftarrow} u_\alpha \overset{\alpha_{1_2}}{\rightarrow} w_1 \overset{\alpha_{2_2}}{\leftarrow} w_1) \overset{\alpha_{1_1}}{\leftarrow} v_\alpha \overset{\alpha_{1_2}}{\rightarrow} w_1$ and $(v_\alpha \overset{\alpha_{1_1}}{\leftarrow} v_1 \overset{\alpha_{1_2}}{\rightarrow} w_1, v_\alpha \overset{\alpha_{1_2}}{\rightarrow} u_\alpha' \overset{\alpha'_{1_2}}{\leftarrow} w_1)$ are bicartesian squares.

We remind that an arrows-only category in the sense of [McL 71] is a category with objects represented by their identities. We remind that a bicartesian square in a category is a diagram $(v_\alpha \overset{\alpha_1}{\leftarrow} u_\alpha \overset{\alpha_2}{\rightarrow} w, v_\alpha' \overset{\alpha_1'}{\leftarrow} u_\alpha' \overset{\alpha_2'}{\rightarrow} w)$ in this category such that $v_\alpha \overset{\alpha_2}{\rightarrow} u_\alpha' \overset{\alpha_1'}{\leftarrow} w$ is a pushout of $v_\alpha \overset{\alpha_1}{\leftarrow} u_\alpha \overset{\alpha_2}{\rightarrow} w$ and $v_\alpha \overset{\alpha_1}{\leftarrow} u_\alpha \overset{\alpha_2}{\rightarrow} w$ is a pullback of $v_\alpha \overset{\alpha_2}{\rightarrow} u_\alpha' \overset{\alpha_1'}{\leftarrow} w$.

The property (A4) is an enriched version of the property which says that $\sigma \tau \sigma = \alpha$ only if $\sigma$ and $\tau$ are identities. The property (A5) is a generalization of Levi Lemma for monoids of words. It means that any two possible decompositions of a process may differ only in a segment that consists of processes which are independent. The property (A6) means that if some processes are independent then their segments are independent too.

The category $\text{Cat}(\text{Proc}(U))$ contains the subcategory $\text{Catg}(\text{Proc}(U))$ of global bounded processes in $U$ and this subcategory also enjoys the properties (A1) - (A6).

### 4 Multiplicative transition systems

Now, if we think of decomposable transitions of multiplicative transition systems as of entities similar to global bounded processes in a universe of objects then we come to the following definition.
4.1. Definition. A multiplicative transition system, or an MTS, is an arrows-only category \(\mathcal{A} = (A, \text{dom}, \text{cod}, ;)\) that enjoys the properties (A1) - (A6).

Morphisms of \(\mathcal{A}\) are called transitions. Identities of \(\mathcal{A}\) are called states. The properties (A1) - (A4) imply the following property.

4.2. Proposition. For every \(\alpha\), the relation \(\sqsubseteq_\alpha\) between decompositions of \(\alpha\) into pairs \((\xi_1, \xi_2)\) such that \(\xi_1\xi_2 = \alpha\), where \((\xi_1, \xi_2) \sqsubseteq_\alpha (\eta_1, \eta_2)\) iff \(\eta_1 = \xi_1\delta\) and \(\xi_2 = \delta\eta_2\) for some \(\delta\), is a partial order.

4.3. Proposition. For every \(\alpha\), the partial order \(\sqsubseteq_\alpha\) between decompositions of \(\alpha\) into pairs \((\xi_1, \xi_2)\) such that \(\xi_1\xi_2 = \alpha\) makes the set of such decompositions a lattice \(LT_\alpha\).

Proof. Let \(\alpha = \xi_1\xi_2 = \eta_1\eta_2\), \(\xi_1 = \sigma_1\alpha_1\), \(\xi_2 = \alpha'_2\sigma_2\), \(\eta_1 = \sigma_1\alpha_2\), \(\eta_2 = \alpha'_1\sigma_2\) with \(\alpha_1, \alpha'_1, \alpha_2, \alpha'_2, \sigma_1, \sigma_2\) as in (A5). The least upper bound of \(x = (\xi_1, \xi_2)\) and \(y = (\eta_1, \eta_2)\) can be defined as \(z = (\xi_1\alpha'_2, \sigma_2) = (\eta_1\alpha'_1, \sigma_2)\). To see this consider any \(u = (\xi_1, \xi_2)\) such that \(x \sqsubseteq_\alpha u\) and \(y \sqsubseteq_\alpha u\). Then \(\zeta_1 = \xi_1\delta\) and \(\zeta_1 = \eta_1\epsilon\) for some \(\delta\) and \(\epsilon\). As \(\alpha'_1\) and \(\alpha'_2\) form a pushout of \(\alpha_1\) and \(\alpha_2\), there exists a unique \(\varphi\) such that \(\delta = \alpha'_2\varphi\) and \(\epsilon = \alpha'_1\varphi\). Hence \(\zeta_1 = \xi_1\alpha'_2\varphi = \eta_1\alpha'_1\varphi\) and, consequently, \(z \sqsubseteq_\alpha u\).

Similarly, due to the fact that \(\alpha_1\) and \(\alpha_2\) form a pullback of \(\alpha'_1\) and \(\alpha'_2\), we obtain that \(t = (\sigma_1, \alpha_1\alpha'_2\sigma_2)\) is the greatest lower bound of \(x\) and \(y\).

4.4. Example. Consider the universe \(U'\) of a producer and a distributor and the concrete processes \(Q, R, S\) in \(U'\) described in example 2.1. By combining the abstract processes corresponding to the possible variants of concrete processes \(Q, R, S\) we obtain a subalgebra \(\mathcal{B}\) of the algebra \(\text{Proc}(U')\) of global bounded processes in \(U'\). This subalgebra is a multiplicative transition system in the sense of definition 4.1.

4.5. Example. Define a transition system without a distinguished initial state as \(M = (S, E, T)\) such that \(S\) is a set of states, \(E\) is a set of events, and \(T \subseteq S \times E \times S\) is a set of transitions, where \((s, e, s') \in T\) stands for the transition from the state \(s\) to the state \(s'\) due to the event \(e\). Assume that \(E\) contains a distinguished element \(*\) standing for "no event", and assume that for every state \(s \in S\) the set \(T\) contains an idle transition \((s, *, s)\) standing for "stay in
s". Then $M$ can be represented by the graph $G(M) = (T, \text{dom}, \text{cod})$, where 
\[ \text{dom}(s, e, s') = (s, *, s) \] and 
\[ \text{cod}(s, e, s') = (s', *, s') \] for every $(s, e, s') \in T$.

Write $s \xrightarrow{\alpha} s'$ to indicate that $(s, e, s') \in T$. Denote by $\text{Lts}$ the set of triples of the form $\alpha = s \xrightarrow{x} s'$ where $x$ is any finite word over the alphabet $E - \{s\}$ such that $x = e_1...e_m$ for $\alpha = s_0 \xrightarrow{e_1} s_1 \xrightarrow{e_2} ... s_{m-1} \xrightarrow{e_m} s_m$ with $s_0 = s$ and $s_m = s'$, or $x$ is the empty word represented by $\ast$ and $s' = s$.

Define $\text{dom}(s \xrightarrow{x} s') = s \xrightarrow{s}$ and $\text{cod}(s \xrightarrow{x} s') = s' \xrightarrow{s'}$.

For triples $\alpha_1 = s_1 \xrightarrow{u_1} s'_1$ and $\alpha_2 = s_2 \xrightarrow{u_2} s'_2$ such that $s'_1 = s_2$ define the result of composing $\alpha_1$ and $\alpha_2$ as $\alpha_1 \alpha_2 = s_1 \xrightarrow{u_1 \alpha_2} s'_2$.

It is easy to verify that the set $\text{Lts}$ with the operations thus defined is a multiplicative transition system $\text{LTS}(M)$ in the sense of definition 4.1. In this system each ordering $\subseteq_\alpha$ is linear and $\langle v \overset{\alpha_1}{\xrightarrow{u_1}} w, v \overset{\alpha_2}{\xrightarrow{u_2}} w' \overset{\alpha'_1}{\xrightarrow{w'}} w \rangle$ is a bicartesian square iff $\alpha_1$ and $\alpha'_1$ are identities or $\alpha_2$ and $\alpha'_2$ are identities. 

4.6. Example. Consider the transition system $M$ from example 4.5. Consider a symmetric irreflexive relation $I \subseteq (E - \{\ast\})^2$, called an independence relation, and the least equivalence relation $\parallel_I$ between words over the alphabet $E - \{\ast\}$ such that words $uabv$ and $ubav$ are equivalent whenever $(a, b) \in I$. The equivalence classes of such a relation are known in the literature as Mazurkiewicz traces with respect to $I$ (see [Maz 88]). Denote by $T$s the set of triples as in example 4.5 but with words over the alphabet $E - \{\ast\}$ replaced by traces with respect to $I$. Define $\text{dom}$ and $\text{cod}$ and the composition as in example 4.5, but with the concatenation of words replaced by the induced concatenation of traces.

It is easy to verify that the set $T$s with the operations thus defined is a multiplicative transition system $\text{TS}(M, I)$ in the sense of definition 4.1, and that this system is a homomorphic image of the system from example 4.5. However, in this system there exist nontrivial bicartesian squares, namely, the squares $\langle v \overset{\alpha_1}{\xrightarrow{u_1}} w, v \overset{\alpha'_2}{\xrightarrow{u'_2}} w \rangle$ such that $\alpha_1 = u \overset{\alpha'_1}{\xrightarrow{v}} w$, $\alpha_2 = u \overset{\alpha'_2}{\xrightarrow{v}} w$, $\alpha'_1 = w \overset{\alpha'_1}{\xrightarrow{u'}} w'$, $\alpha'_2 = v \overset{\alpha'}{\xrightarrow{u}} w'$ with $(a, b) \in I$ for all $(a, b)$ such that $a$ occurs in $x_1$ and $b$ occurs in $x_2$. 

\[ \square \]
5 Independence and equivalence of transitions

In [Wink 03] and [Wink 09b] it has been shown that the natural concepts of sequential and parallel independence of finitary processes of a condition/event Petri net in the sense of [Petri 80], similar to the concepts introduced in [EK 76], can be characterized in the category of such processes as the existence of appropriate bicartesian squares. Now we shall use this characterization to define independence and a natural equivalence of elements of multiplicative transition systems similar to the considered in [WN 95] independence and equivalence of transitions in transition systems with independence. This will allow us to adapt and study the concept of a region similar to that introduced in [ER 90].

5.1. Definition. If \((v \xrightarrow{\alpha_1} u \xrightarrow{\alpha_2} w, v \xrightarrow{\alpha_2'} u' \xrightarrow{\alpha_1'} w)\) is a bicartesian square in a multiplicative transition system \(A = (A, \text{dom}, \text{cod};)\) then we say that the transitions \(u \xrightarrow{\alpha_1} v\) and \(u \xrightarrow{\alpha_2} w\) are parallel independent, and the transitions \(u \xrightarrow{\alpha_1} v\) and \(v \xrightarrow{\alpha_2'} u'\), as well as the transitions \(u \xrightarrow{\alpha_2} w\) and \(w \xrightarrow{\alpha_1'} u'\), are sequential independent. ♯

5.2. Examples. In the multiplicative transition system \(B\) in example 4.4 transitions \(\pi + \text{dom}(\rho)\) and \(\text{dom}(\pi) + \rho\) are parallel independent, transitions \(\pi + \text{dom}(\rho)\) and \(\text{cod}(\pi) + \rho\) are sequential independent, and transitions \(\text{dom}(\pi) + \rho\) and \(\pi + \text{cod}(\rho)\) are sequential independent. In the multiplicative transition system \(\text{LTS}(M)\) in example 4.5 transitions \(u \xrightarrow{\alpha_1} v\) and \(u \xrightarrow{\alpha_2} w\) are parallel independent only if one of them is an identity. Similarly, transitions \(u \xrightarrow{\alpha_1} v\) and \(v \xrightarrow{\alpha_2'} u'\) are sequential independent only if one of them is an identity. In the multiplicative transition system \(\text{TS}(M)\) in example 4.6 transitions \(u \xrightarrow{\alpha_1} v\) and \(u \xrightarrow{\alpha_2} w\) are parallel independent iff \((a, b) \in I\) for all \(a\) occurring in \(\alpha_1\) and all \(b\) occurring in \(\alpha_2\). Similarly, transitions \(u \xrightarrow{\alpha_1} v\) and \(v \xrightarrow{\alpha_2} u'\) are sequential independent iff \((a, b) \in I\) for all \((a, b)\) such that \(a\) occurs in \(\alpha_1\) and \(b\) occurs in \(\alpha_2\). ♯

5.3. Definition. By the natural equivalence of elements of a multiplicative transition system \(A = (A, \text{dom}, \text{cod};)\) we mean the least equivalence relation \(\equiv\) in \(A\) such that \(\alpha_1 \equiv \alpha'_1\) whenever in this system there exists a bicartesian square \((v \xrightarrow{\alpha_1} u \xrightarrow{\alpha_3} w, v \xrightarrow{\alpha_3'} u' \xrightarrow{\alpha'_1} w)\). ♯
5.4. Examples. In the multiplicative transition system $\mathcal{B}$ in example 4.4 transitions $\pi + \text{dom}(\rho)$ and $\text{cod}(\rho) + \pi$ are equivalent in the sense of definition 5.3. In the multiplicative transition system $\text{LTS}(M)$ in example 4.5 the natural equivalence coincides with the identity relation. In the multiplicative transition system $\text{TS}(M)$ in example 4.6 we have $\alpha_1 \equiv \alpha_1'$ whenever $(v \overset{\alpha_1}{\rightarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_1'}{\rightarrow} w)$ with $\alpha_1$ and $\alpha_1'$ representing the same trace $t_1$, and $\alpha_2$ and $\alpha_2'$ representing the same trace $t_2$, for $(a,b) \in I$ for all $(a,b)$ such that $a$ occurs in $t_1$ and $b$ occurs in $t_2$. 

6 Regions

The existence in multiplicative transition systems of the natural equivalence transitions allows us to adapt and exploit the concept of a region similar to that introduced in [ER 90].

6.1. Definition. By a region of a multiplicative transition system $\mathcal{A} = (A, \text{dom}, \text{cod}, ;)$ we mean a nonempty subset $r$ of the set of states of $\mathcal{A}$ such that:

\[
\begin{align*}
\text{dom}(\alpha) &\in r \text{ and } \text{cod}(\alpha) \notin r \text{ and } \alpha' \equiv \alpha \\
\text{dom}(\alpha) &\notin r \text{ and } \text{cod}(\alpha) \in r \text{ and } \alpha' \equiv \alpha \\
\text{implies } \text{dom}(\alpha') &\in r \text{ and } \text{cod}(\alpha') \notin r, \\
\text{dom}(\alpha) &\notin r \text{ and } \text{cod}(\alpha) \in r \text{ and } \alpha' \equiv \alpha \\
\text{implies } \text{dom}(\alpha') &\notin r \text{ and } \text{cod}(\alpha') \in r.
\end{align*}
\]

6.2. Example. Consider the multiplicative transition system $\mathcal{B}$ in example 4.4. In this system the sets $[(p,q)] = \{(p,q)\} \times \{(d) \times [0, +\infty)\}$ with $q \geq 0$, the sets $[(d,r)] = \{(d,r)\} \times \{(p) \times [0, +\infty)\}$ with $r \geq 0$, and disjoint unions of such sets are regions.

6.3. Example. Consider the transition system $M'$ in figure 6.1. Consider the independence relation $I' = \{(a,b), (a_1,b_1), (a,b), (a_1,b_1)\}$ and the multiplicative transition system $\text{TS}(M', I')$. In this system we have transitions $\alpha = u \overset{[a]}{\rightarrow} v, \beta = u \overset{[b]}{\rightarrow} w, \alpha' = u \overset{[a]}{\rightarrow} u', \beta' = v \overset{[b]}{\rightarrow} u' \overset{[a]}{\rightarrow} w', \beta'' = z \overset{[b]}{\rightarrow} b', \alpha_1 = u \overset{[a_1]}{\rightarrow} v', \beta_1 = u \overset{[b_1]}{\rightarrow} w', \alpha_1' = u \overset{[a_1]}{\rightarrow} u, \beta_1' = v \overset{[b_1]}{\rightarrow} u \overset{[a_1]}{\rightarrow} v \overset{[a_1]}{\rightarrow} z, \beta_1'' = w \overset{[b_1]}{\rightarrow} t$, where $[a],[a_1],[b],[b_1]$ are traces corresponding to $a, a_1, b, b_1$, and compositions of these transitions. For example, $\alpha \beta' = \beta \alpha' = \gamma = u \overset{[ab]}{\rightarrow} u'$.
\[ \alpha_1 \beta_1' = \beta_1 \alpha_1' = \gamma_1 = u' \xrightarrow{[a_1b_1]} u, \text{ transitions } \alpha, \alpha' \text{ are equivalent, transitions } \beta, \beta' \text{ are equivalent, and we have regions } \\
E = \{u, w, t, v', z\}, \ F = \{u, v, z, t, w'\}, \ G = \{v, u', w'\}, \ H = \{w, u', v'\}, \ E \cup G, \ F \cup H, \text{ and } \{u, v, w, z, t, u', v', w'\}. \]

\[ \xymatrix{ 
& t 
& w' \\
\ast 
& a 
& w' \\
b_1 
& a 
& b_1 \\
\ast 
& b 
& v \\
b 
& a 
& b \\
u 
& a 
& v \\
& b 
& b_1 \\
& \ast \\
& z \\
} \]

\[ M' \]

From the definition of a region we obtain the following proposition.

6.4. Proposition. If \( \mathcal{A} = (A, \text{dom}, \text{cod},; ) \) is a multiplicative transition system, \( r \) is a region of \( \mathcal{A} \), and \( (v \xleftarrow{a_1} u \xrightarrow{a_2} w, v \xleftarrow{a_2'} u' \xrightarrow{a_1'} w) \) is a bicartesian square in \( \mathcal{A} \), then \( v \in r \) implies that \( u \in r \) or \( u' \in r \). \( \sharp \)

Due to the property (A6) of multiplicative transition systems we obtain the following proposition.

6.5. Proposition. If \( \mathcal{A} = (A, \text{dom}, \text{cod},; ) \) is a multiplicative transition system, \( r \) is a region of \( \mathcal{A} \), and \( (v \xleftarrow{a_1} u \xrightarrow{a_2} w, v \xleftarrow{a_2'} u' \xrightarrow{a_1'} w) \) is a bicartesian square

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in $\mathcal{A}$ with morphisms which are not identities, then for every decomposition $u \xrightarrow{\alpha} v = u \xrightarrow{\alpha_1} v_1 \xrightarrow{\alpha_2} v$ such that $u, v \in r$ we have $v_1 \in r$, and for every decomposition $w \xrightarrow{\alpha'} u' = w \xrightarrow{\alpha'_{11}} w_1 \xrightarrow{\alpha'_{12}} u'$ such that $w, u' \in r$ we have $w_1 \in r$.  

\[\sharp\]

The following three propositions follow from the definition of a region.

6.6. Proposition. The set of all states of $\mathcal{A}$ is a region of $\mathcal{A}$.  

6.7. Proposition. If $p$ and $q$ are disjoint regions of $\mathcal{A}$ then $p \cup q$ is a region of $\mathcal{A}$.  

6.8. Proposition. If $p$ and $q$ are different regions of $\mathcal{A}$ such that $p \subseteq q$ then $q - p$ is a region of $\mathcal{A}$.  

Given a chain $(r_i : i \in I)$ of regions with $r = \bigcap (r_i : i \in I)$ and a transition $\alpha$ such that $\text{dom}(\alpha) \in r$ and $\text{cod}(\alpha) \notin r$, there exists $i_0 \in I$ such that $\text{dom}(\alpha) \in r_i$ and $\text{cod}(\alpha) \notin r_i$ for $i > i_0$. Consequently, for every transition $\alpha'$ such that $\alpha' \equiv \alpha$ we have $\text{dom}(\alpha) \in r_i$ and $\text{cod}(\alpha) \notin r_i$ for $i > i_0$, and thus $\text{dom}(\alpha) \in r$ and $\text{cod}(\alpha) \notin r$. Similarly, for $\alpha$ such that $\text{dom}(\alpha) \notin r$ and $\text{cod}(\alpha) \in r$ and for $\alpha' \equiv \alpha$. So, $r$ is a region. Hence, taking into account Kuratowski - Zorn Lemma, we obtain the following results.

6.9. Proposition. Every region of $\mathcal{A}$ contains a minimal region.  

The propositions 6.8 and 6.9 imply the following properties.

6.10. Proposition. Every state of $\mathcal{A}$ belongs to a minimal region.  

6.11. Proposition. If a state $s$ of $\mathcal{A}$ does not belong to a region $r$ then there exists a minimal region $r'$ such that $r \cap r' = \emptyset$ and $s$ belongs to $r'$.  

6.12. Proposition. Every region of $\mathcal{A}$ can be represented as a disjoint union of minimal regions.  

\[\sharp\]
7 Transitions as labelled posets

Now we shall concentrate on multiplicative transition systems which enjoy a specific but still very natural property. We shall call them clean multiplicative transition systems, and we shall show that their elements can be interpreted as processes in a universe of objects.

We start with suitable notions and observations.

Let \( A = (A, \text{dom}, \text{cod}, ;) \) be a multiplicative transition system.

7.1. Definition. Given \( \alpha \in A \), by a cut of \( \alpha \) we mean a pair \( x = (\xi_1, \xi_2) \) such that \( \xi_1 \xi_2 = \alpha \), by a state corresponding to such a cut \( x \) we mean \( \text{cod}(\xi_1) \), and we write such a state as \( \text{state}_\alpha(x) \).

For every transition \( \alpha \in A \) we have the relation \( \sqsubseteq_\alpha \), where \( x \sqsubseteq_\alpha y \) with \( x = (\xi_1, \xi_2) \) and \( y = (\eta_1, \eta_2) \) means that \( \eta_1 = \xi_1 \delta \) with some \( \delta \). From proposition 4.2 it follows that \( \sqsubseteq_\alpha \) is a partial order, and that for \( x = (\xi_1, \xi_2) \) and \( y = (\eta_1, \eta_2) \) such that \( x \sqsubseteq_\alpha y \) there exists a unique \( \delta \) such that \( \eta_1 = \xi_1 \delta \), written as \( x \rightarrow_\alpha y \).

From proposition 4.3 it follows also that this partial order makes the set of cuts of \( \alpha \) a lattice \( \text{LT}_\alpha \). It is easy to see that the lattice \( \text{LT}_\alpha \) viewed as a category is a multiplicative transition system and that the obvious extension of the correspondence \( x \mapsto \text{state}_\alpha(x) \) to the mapping \( \text{mp}_\alpha \) from \( \text{LT}_\alpha \) to \( A \) preserves the composition. Given two cuts \( x \) and \( y \), by \( x \sqcup_\alpha y \) and \( x \sqcap_\alpha y \) we denote respectively the least upper bound and the greatest lower bound of \( x \) and \( y \). The diagram \( (x \leftarrow x \sqcap_\alpha y \rightarrow y, x \rightarrow x \sqcup_\alpha y \leftarrow y) \) is a bicartesian square in \( \text{LT}_\alpha \). From (A5) it follows that the image under the mapping \( \text{mp}_\alpha \) of such a diagram is a bicartesian square in \( A \).

7.2. Example. Consider the multiplicative transition system \( B \) in example 4.4. For the transition \( \tau = [T] = \sigma'(\pi + \rho)\sigma'' \) of this system described in examples 2.1 and 3.1 we have the multiplicative transition system \( \text{LT}_\tau \) shown in figure 7.1 and its minimal regions

\[ i = \{(u, \tau)\}, \]
\[ j = \{\sigma'(\pi + \rho)\sigma''\}, ..., (\sigma'(\pi + \dom(\rho)) (\text{cod}(\pi) + \rho)\sigma'')\}, ..., \]
\[ j' = \{\sigma' (\text{dom}(\pi) + \rho), (\pi + \text{cod}(\rho))\sigma'', ..., (\sigma' + \rho, \sigma'')\}, ... \]
\[ k = \{\sigma' (\pi + \rho)\sigma''), ..., (\sigma' (\text{dom}(\pi) + \rho), (\pi + \text{cod}(\rho))\sigma'')\}, ... \]
\[ k' = \{\sigma' (\pi + \text{dom}(\rho)), (\text{cod}(\pi) + \rho)\sigma'', ..., (\sigma' + \rho, \sigma'')\}, \]
\[ l = \{(\tau, u)\}. \]
7.3. Example. Consider the multiplicative transition system $TS(M', I')$ in example 6.3. For the transition $\delta = \gamma \gamma_1 = \alpha \beta' \alpha_1 \beta_1'$ of this system we have the multiplicative transition system $LT_\delta$ shown in figure 7.2 and its minimal regions $e = \{(u, \delta), (\beta, \alpha' \gamma_1), (\beta \beta_2', \alpha'' \alpha_1')\}$, $g = \{\{(\alpha, \beta' \gamma_1), (\gamma, \gamma_1), (\gamma \beta_1, \alpha_1')\}\}$, $e' = \{\{(\alpha \alpha_2', \beta'' \beta_1'), (\gamma \alpha_1, \beta_1'), (\delta, u)\}\}$, $f = \{(u, \delta), (\alpha, \beta' \gamma_1), (\alpha \alpha_2', \beta'' \beta_1')\}$, $h = \{\{(\beta, \alpha' \gamma_1), (\gamma, \gamma_1), (\gamma \alpha_1, \beta_1')\}\}$, $f' = \{\{(\beta \beta_2', \alpha'' \alpha_1'), (\gamma \beta_1, \alpha_1')\}\}$. 

$$
\begin{array}{c}
\sigma'(\text{dom}(\pi) + \rho), (\pi + \text{cod}(\rho)) \sigma'') \rightarrow \ldots \rightarrow \sigma'(\pi + \rho) \sigma'' \rightarrow (\tau, u)
\end{array}
$$

$$
\begin{array}{c}
(u, \tau) \rightarrow (\sigma', (\pi + \rho) \sigma'') \rightarrow \ldots \rightarrow (\sigma'(\pi + \text{dom}(\rho)), (\text{cod}(\pi) + \rho) \sigma'')
\end{array}
$$

$LT_\tau$

Figure 7.1

$LT_\delta$

Figure 7.2
Let \( \mathcal{A} = (A, \text{dom}, \text{cod}, \cdot) \) be an arbitrary multiplicative transition system.

Given an element \( \alpha \) of \( \mathcal{A} \), by \( R_\alpha \) we denote the set of minimal regions of the multiplicative transition system \( \text{LT}_\alpha \).

Using regions of \( \mathcal{A} \) we want to assign to each transition \( \alpha \) of \( \mathcal{A} \) a labelled partially ordered set (an lposet) \( \text{L}_\alpha = (X_\alpha, \leq_\alpha, l_\alpha) \). Each element \( x \in X_\alpha \) is supposed to play the role of an occurrence in \( \alpha \) of a minimal region \( l_\alpha(x) \) of \( \mathcal{A} \). The partial order \( \leq_\alpha \) is supposed to reflect how occurrences of minimal regions arise from other minimal occurrences.

The underlying set \( X_\alpha \) of \( \text{L}_\alpha \) is supposed to be defined referring to the set \( R_\alpha \) of minimal regions of the multiplicative transition system \( \text{LT}_\alpha \) and to a relation \( \vdash_\alpha \) between minimal regions of \( \text{LT}_\alpha \) and minimal regions of \( \mathcal{A} \).

We are going to show how to define the respective lposet \( \text{L}_\alpha \) for every element of \( \mathcal{A} \) if \( \mathcal{A} \) is clean in the following sense.

**7.4. Definition.** The multiplicative transition system \( \mathcal{A} \) is said to be clean if for every element \( \alpha \) of this system every minimal region \( r \in R_\alpha \) is convex in the sense that \( w \in r \) for every \( w \) such that \( u \vdash_\alpha w \vdash_\alpha v \) for some \( u \in r \) and \( v \in r \).

If the MTS \( \mathcal{A} \) is clean then in the set \( R_\alpha \) there exists a partial order that can be defined as follows.

**7.5. Definition.** Given \( x, y \in R_\alpha \), we write \( x \preceq_\alpha y \) iff for every \( v \in y \) there exists \( u \in x \) such that \( u \subseteq_\alpha v \), for every \( u \in x \) there exists \( v \in y \) such that \( u \subseteq_\alpha v \), and the following conditions are satisfied:

1. \( t \in x \) iff \( w \in y \), for every bicartesian square \( (u \leftarrow t \rightarrow w, u \rightarrow v \leftarrow w) \) with \( u \in x \) and \( v \in y \).
2. \( t' \in x \) iff \( w' \in y \), for every bicartesian square \( (t' \leftarrow u \rightarrow v, t' \rightarrow w' \leftarrow v) \) with \( u \in x \) and \( v \in y \).

**7.6. Proposition.** If \( x, y \in R_\alpha \) are not disjoint and different then neither \( x \preceq_\alpha y \) nor \( y \preceq_\alpha x \).

**Proof.** Suppose that \( x \) and \( y \) are different minimal regions of \( \text{LT}_\alpha \) such that \( x \cap y \neq \emptyset \). Then \( x - y \) and \( y - x \) are nonempty and there exist \( u \in x - y \), \( v \in y - x \).
\( v \in y - x, \) and \( w, z \in x \cap y \) such that \( u \) and \( w \) are adjacent nodes of a bicartesian square \( U, \) \( z \) and \( v \) are adjacent nodes of a bicartesian square \( V, \) and the nodes of the bicartesian square \( W = (w \leftarrow w \cap z \rightarrow z, w \rightarrow w \sqcup z \leftarrow z) \) are in \( x \cap y. \)

Consider the case in which \( w = u \sqcup u' \) for some \( u' \) not in \( x \) and \( z = v \sqcap v' \) for some \( v' \) not in \( y, \) as it is depicted in figure 7.3. Then \( u' \in y, v' \in x, \) and the condition (1) is not satisfied for \( z \sqsubseteq v \) and the bicartesian square \( (v \leftarrow z \rightarrow v', v \rightarrow v \sqcup v' \leftarrow v'). \) Consequently, \( x \not\leq y \) does not hold.

Similarly, in the other possible cases we come to the conclusion that neither \( x \not\leq y \) nor \( y \not\leq x.\)

\[\text{Figure 7.3}\]

**7.7. Proposition.** If the MTS \( \mathcal{A} \) is clean and \( x, y \in R_\alpha \) are disjoint then either \( x \leq y \) or \( y \leq x.\)

**Proof.** It is impossible that \( u \) and \( v \) are incomparable for all \( u \in x \) and \( v \in y \) since one of the regions \( x \) or \( y \) contains \( u \sqcup v \) or \( u \sqcap v. \)

Suppose that \( u \sqsubseteq v \) for \( u \in x \) and \( v \in y. \) As \( x \) and \( y \) are disjoint and convex, it suffices to prove that every element of \( y \) has a predecessor in \( x.\)
Consider $w \in y$. If $v \sqsubseteq w$, then $u \sqsubseteq w$. If $w \sqsubseteq v$, then $u' \sqsubseteq w$ for $u' = w \cap w$ and by considering the bicartesian square $(u' \leftarrow u \rightarrow v, u' \rightarrow w \leftarrow v)$, we obtain that $u' \in x$. If $w$ and $v$ are incomparable then either $v \cap w \in y$ and we may replace $w$ by $v \cap w$ and proceed as in the previous case, or $v \cup w \in y$ and by considering the bicartesian square $(u' \leftarrow u \rightarrow w, u \rightarrow v \cup w \leftarrow v)$, we obtain that $u' \sqsubseteq w$ for $u' \in x$. On the other hand, $u \sqsubseteq v$, for $u \in x$ and $v \in y$ exclude $v' \sqsubseteq u'$ for $u' \in x$ and $v' \in y$ since $x$ and $y$ are convex. Hence $x \sqsubseteq w$.

Similarly, in the case $v \sqsubseteq u$, we obtain $y \sqsubseteq x$.

7.8. Proposition. If the MTS $A$ is clean then the relation $\leq$ is a partial order on $R_A$.

Proof. The transitivity of the relation $\leq$ follows from the definition of this relation. The antisymmetry follows from the transitivity and from the propositions 7.6 and 7.7.

The relation $\vdash$ between minimal regions of $LT_A$ and minimal regions of $A$ can be defined as follows.

7.9. Proposition. For every minimal region $m$ of $LT_A$, there exists a minimal region $r$ of $A$ such that the set $\text{state}_\alpha(m) = \{\text{state}_\alpha(u) : u \in m\}$ is contained in $r$, and we write $m \vdash r$. \qed

Proof. Given a minimal region $m$ of $LT_A$, let $r$ be a minimal element of the set of regions of $A$ containing the set $\text{state}_\alpha(m)$. As the image of every bicartesian square of $LT_A$ under the mapping $mp_\alpha$ from $LT_A$ to $A$ is a bicartesian square in $A$, and for every partition of $m$ into two disjoint nonempty subsets $m'$ and $m''$, there exists in $LT_A$ a bicartesian square connecting $m'$ and $m''$, the same holds true for $r$. Consequently, $r$ is a minimal region of $A$. \qed

Finally, the lposet $L_\alpha = (X_\alpha, \leq, l_\alpha)$ can be defined by defining $X_\alpha$ as the set of pairs $(m, r)$ such that $m \in R_\alpha$ and $m \vdash r$, the relation $\leq$ as the partial order on $X_\alpha$ such that $x \leq x'$ for $x = (m, r)$ and $x' = (m', r')$ whenever $m \leq m'$, and $l_\alpha(x)$ as $r$ for $x = (m, r) \in X_\alpha$. 21
7.10. Example. Consider the multiplicative transition system $B$ described in example 4.4, its minimal regions $[(p, q), [(d, r)]$ described in example 6.2, and the minimal regions $i, j, k, k', l$ of $LT_\tau$ for $\tau = [T] = \sigma'(\pi + \rho)\sigma''$ as in example 7.2. We obtain $L_\tau = (X_\tau, \leq_\tau, l_\tau)$, where

$$X_\tau = \{ (i, [(p, q_0 + m)]), (i, [(d, r_0 - m)]), (j, [(p, q_0)]), ..., (j', [(p, q_1)]), (k, [(d, r_0)]), ..., (k', [(d, r_1)]), (l, [(p, q_1 - m')]), (l, [(d, r_1 + m')]) \},$$

$$\leq_\tau (i, [(p, q_0 + m)]), (i, [(d, r_0 - m)]) \leq_\tau \{ (j, [(p, q_0)]), \leq_\tau (j', [(p, q_1)]), \leq_\tau (k, [(d, r_0)]), \leq_\tau (k', [(d, r_1)]), \leq_\tau (l, [(p, q_1 - m')]), \leq_\tau (l, [(d, r_1 + m')]),$$

$$l_\tau((i, [(p, q_0 + m)])) = [(p, q_0 + m)], l_\tau((j, [(p, q_0)])) = [(p, q_0)], l_\tau((j', [(p, q_1)])) = [(p, q_1)], l_\tau((k, [(d, r_0)])) = [(d, r_0)], l_\tau((k', [(d, r_1)])) = [(d, r_1)], l_\tau((l, [(p, q_1 - m')])) = [(p, q_1 - m')], l_\tau((l, [(d, r_1 + m']))) = [(d, r_1 + m')].$$

The corresponding $[L_\tau]$ is essentially as that in figure 2.1.

7.11. Example. Consider the BOPC $TS(M', I')$ described in example 6.3, its minimal regions $E, F, G, H$, and the minimal regions $e, g, e', f, h, f'$ of $LT_\delta$ for $\delta = \gamma_1 = \alpha\beta\alpha_1\beta_1'$ as in example 7.3. We obtain $L_\delta = (X_\delta, \leq_\delta, l_\delta)$, where $X_\delta = \{ (e, E), (g, G), (e', E), (f, F), (h, H), (f', F) \}, (e, E) \leq_\delta (g, G) \leq_\delta (e', E), (f, F) \leq_\delta (h, H) \leq_\delta (f', F), l_\delta((e, E)) = l_\delta((e', E)) = E, l_\delta((g, G)) = G, l_\delta((f, F)) = l_\delta((f', F)) = F, l_\delta((h, H)) = H$. The corresponding $[L_\delta]$ is presented in figure 7.4.

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7.12. Proposition. If the MTS $A$ is clean then for every element $u$ of $LT_\alpha$, and for every $x, y \in R_\alpha$ such that $x \preceq_\alpha y$, and $x \preceq_\alpha x'$ for some $x' \in X_\alpha$ such that $u \in x'$, and $y' \preceq_\alpha y$ for some $y' \in X_\alpha$ such that $u \in y'$, there exists $z \in X_\alpha$ such that $u \in z$, and $x \preceq_\alpha z$, and $z \preceq_\alpha y$.

Proof. For $x' = x$ it suffices to define $z$ as $x$. For $y' = y$ it suffices to define $z$ as $y$. Consider the case in which $x' \neq x$ and $y' \neq y$. By proposition 7.6 in this case $x$ and $y$ are disjoint, $x'$ and $x$ are disjoint, and $y'$ and $y$ are disjoint. Consequently, $u$ does not belong to $x$, $u$ does not belong to $y$, and, by proposition 6.11, there exists $z \in X_\alpha$ that is disjoint both with $x$ and with $y$, as required.

Crucial for a representation of multiplicative transition systems are the properties of $A$ described in proposition 7.12 and in the following propositions.

7.13. Proposition. If the MTS $A$ is clean then every two different minimal regions $x$ and $y$ of $LT_\alpha$ such that $x \vdash_\alpha r$ and $y \vdash_\alpha r$ for a minimal region $r$ of $A$ are disjoint.
Proof. The correspondence between $u \delta \rightarrow v$ such that $u = (\xi_1, \xi_2)$, $v = (\eta_1, \eta_2)$, $\eta_1 = \xi_1 \delta$, $\xi_2 = \delta \eta_2$ and $mp_\alpha(u) \delta \rightarrow mp_\alpha(v)$ is a functor $F_\alpha$ from $LT_\alpha$ to $\mathcal{A}$. Due to (A5) this functor preserves bicartesian squares. Consequently, $mp_\alpha^{-1}(r)$ is a region in $LT_\alpha$.

Say that elements $a, b \in mp_\alpha^{-1}(r)$ are connected if in $LT_\alpha$ there exists a bicartesian square $S$ with one side with the vertices $a$ and $b$ and with the opposite side with the images of vertices under $mp_\alpha$ not in $r$. The reflexive and transitive closure of the respective connection relation in $mp_\alpha^{-1}(r)$ is an equivalence and divides $mp_\alpha^{-1}(r)$ into a family $D$ of disjoint components. Some of these components can be non-separable in the sense that they contain elements $a$ and $b$ such that $a \in s$ iff $b \in s$ for every region $s \subseteq mp_\alpha^{-1}(r)$. The reflexive and transitive closure of this relation divides $D$ into a family of equivalence classes with unions being minimal regions of $LT_\alpha$. The minimal regions thus obtained form a unique decomposition of $mp_\alpha^{-1}(r)$ into a disjoint union of minimal regions. As $x$ and $y$ are different minimal regions contained in $mp_\alpha^{-1}(r)$, they must be different elements of this unique decomposition. Consequently, they must be disjoint.

7.14. Proposition. If the MTS $\mathcal{A}$ is clean then for every $\alpha$ in $\mathcal{A}$ and for $x, y \in X_\alpha$, the equality $l_\alpha(x) = l_\alpha(y)$ implies $x \leq_\alpha y$ or $y \leq_\alpha x$.

Proof. It suffices to take into account propositions 7.6 and 7.12.

8 Towards a representation

The construction of the labelled poset $L_\alpha = (X_\alpha, \leq_\alpha, l_\alpha)$ for every element $\alpha$ of a clean MTS $\mathcal{A}$ is such that due to the properties (A1) - (A4) of $\mathcal{A}$ we obtain that no segment of $L_\alpha$ is isomorphic to its subsegment. This suggests that elements of clean MTSs represent processes in a universe of objects.

To see this, consider the universe $U(\mathcal{A}) = (W(\mathcal{A}), V(\mathcal{A}), ob(\mathcal{A}))$ of objects, where $V(\mathcal{A})$ is the set of decompositions of the set of states of $\mathcal{A}$ into disjoint unions of minimal regions of $\mathcal{A}$, $W(\mathcal{A})$ is the set of pairs $w = (v, r)$ consisting of a decomposition $v$ of the set of states of $\mathcal{A}$ into a disjoint union of minimal regions of $\mathcal{A}$ and of a minimal region $r \in v$, and $(ob(\mathcal{A}))(w) = v$ for every $w = (v, r) \in W(\mathcal{A})$. Due to proposition 6.12 the sets $V(\mathcal{A})$ and $W(\mathcal{A})$ are nonempty.
Given $\alpha \in A$, consider the lposet $L^*_\alpha = (X^*_\alpha, \leq^*_\alpha, l^*_\alpha)$, where $X^*_\alpha$ is the set of triples $(m, v, r)$ such that such that $m \in R_\alpha$ and $m \vdash_\alpha r$ and $(v, r) \in W(A)$, the relation $\leq^*_\alpha$ is the partial order on $X^*_\alpha$ such that $x \leq^*_\alpha x'$ for $x = (m, r, v)$ and $x' = (m', r', v')$ whenever $m \leq^*_\alpha m'$, and $v = v'$ if $m = m'$ and $r = r'$, and $l^*_\alpha(x) = (v, r)$ for $x = (m, r, v) \in X^*_\alpha$.

As the minimal regions of every decomposition $v \in V(A)$ are disjoint, due to proposition 7.6 we obtain easily that the set $X^*_\alpha|v = \{ x \in X^*_\alpha : (ob(A))(l^*_\alpha(x)) = v \}$ is a maximal chain and has an element in every cross-section of $L^*_\alpha$. As also every element of $X^*_\alpha$ belongs to a cross-section of $L^*_\alpha$, we obtain that $L^*_\alpha$ is a concrete process in $U(A)$. Consequently, we obtain the following proposition.

8.1. Proposition. Given a clean MTS $A$, the correspondence

$$\alpha \mapsto [L^*_\alpha] = [(X^*_\alpha, \leq^*_\alpha, l^*_\alpha)]$$

between elements of $A$ and pomsets is a mapping from $A$ to the category of processes in the universe $U(A) = (W(A), V(A), ob(A))$ of objects $\mathcal{A}$. 

8.2. Example. Consider the MTS represented by the diagram in figure 8.1, where $\alpha\beta' = \beta\alpha'$, $\alpha'\gamma' = \gamma\alpha''$, $\delta\gamma'' = \gamma'\delta'$. In this system the diagrams $(v \overset{\beta}{\rightarrow} u \overset{\beta'}{\rightarrow} w, v \overset{\beta'}{\rightarrow} u' \overset{\beta'}{\rightarrow} w), (u' \overset{\alpha'}{\leftarrow} w \overset{\gamma}{\rightarrow} \pi, u' \overset{\gamma'}{\leftarrow} z \overset{\alpha''}{\rightarrow} \pi), (t \overset{\delta}{\leftarrow} u' \overset{\gamma'}{\rightarrow} z, t \overset{\gamma''}{\leftarrow} u'' \overset{\delta'}{\rightarrow} z)$ are cartesian squares, the sets $uv\pi = \{ u, w, \pi \}$, $uv'z = \{ v, u', z \}$, $tu'' = \{ t, u'' \}$, $wu'z = \{ w, u', \pi, z \}$, $wu = \{ u, v \}$, $wu't = \{ w, u', t \}$, $\pi zu'' = \{ \pi, z, u'' \}$ are minimal regions, and we have the following decompositions of the set of states into disjoint unions of minimal regions

$I = \{ uv\pi, vu'z, tu'' \}$, $J = \{ uv, wu'\pi z, tu'' \}$, $K = \{ uv, wu't, \pi zu'' \}$. Consequently, the respective universe of objects is $U' = (W', V', ob')$, where

$V' = \{ I, J, K \}$,

$W' = \{ (I, uv\pi), (I, vu'z), (I, tu''), (J, uv), (J, wu'\pi z), (J, tu''), (K, uv), (K wu't), (K, \pi zu'') \}$,

$ob'(I, uv\pi) = ob'(I, vu'z) = ob'(I, tu'') = I$,

$ob'(J, uv) = ob'(J, wu'\pi z) = ob'(J, tu'') = J$,

$ob'(K, uv) = ob'(K, wu't) = ob'(K, \pi zu'') = K$. 

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Consider the transition \( \pi = \alpha \beta \delta \gamma'' \) of this system. The lattice \( LT_\pi \) of decompositions of this transition is essentially identical with the system itself, and we have the following set of minimal regions of this lattice

\[
R_\pi = \{uw\pi, vu'z, tu'', uv, wu'\pi z, wu't, \pi z u''\},
\]

where

\[
uw\pi \preceq_\pi vu'z \preceq_\pi tu'', uv \preceq_\pi wu'\pi z \preceq_\pi tu'', wv \preceq_\pi wu't \preceq_\pi \pi z u''.
\]

Consequently,

\[
X^*_\pi = \{(uw\pi, I, uw\pi), (vu'z, I, vu'z)(tu'', I, tu''), (uv, J, uv),
(wu'\pi z, J, wu'\pi z), (tu'', J, tu''), (uv, K, uv),
(wu't, K, wu't), (\pi z u'', K, \pi z u'')\}
\]

with the partial order \( \leq_\pi \) induced by \( \preceq_\pi \), and we obtain the process in \( U' \) shown in figure 8.2.  

\[\uparrow\]
Note that the correspondence \( \alpha \mapsto [L^*_{\alpha}] = [(X^*_\alpha, \leq^*_\alpha, l^*_\alpha)] \) need not be a homomorphism. To see this it suffices to consider a clean MTS \( A \) that is the reduct of an algebra of processes, and in this MTS a process \( \gamma = \alpha \beta \), where \( \alpha = \text{dom}(\varphi) + \psi \) and \( \beta = \varphi + \text{cod}(\psi) \). It is easy to see that \([L^*_{\alpha}] \neq [L^*_{\alpha}] [L^*_{\beta}]\).

However, every process \( L^*_{\alpha} \) can be transformed into a process \( L^{**}_{\alpha} \) such that the correspondence \( \alpha \mapsto [L^{**}_{\alpha}] \) is a homomorphism. This can be done as follows.

The fact that all \((m,r,v) \in X^*_\alpha\) with the same \( r \) and \( v \) form a chain implies the following proposition.

**8.3. Proposition.** The following relation between elements of \( X^*_\alpha \) is an equivalence relation: \((m,r,v) \simeq (m',r',v')\) iff \( v' = v \), \( r' = r \), \( m \vdash_{\alpha} r \), \( m' \vdash_{\alpha} r \), and \( m'' \vdash_{\alpha} r \) for all \( m'' \) such that \( m \sqsubseteq_{\alpha} m'' \sqsubseteq_{\alpha} m' \) or \( m' \sqsubseteq_{\alpha} m'' \sqsubseteq_{\alpha} m \). 

Due to this proposition it is straightforward to prove the following proposition.

**8.4. Proposition.** The triple \( L^{**}_{\alpha} = (X^{**}_{\alpha}, \leq^{**}_{\alpha}, l^{**}_{\alpha}) \) with \( X^{**}_{\alpha} = X^*_\alpha / \simeq_{\alpha} \), \( x \leq^{**}_{\alpha} x' \) whenever \((m,r,v) \leq^*_\alpha (m',r',v')\) for some \((m,r,v) \in x\) and \((m',r',v') \in x'\), and \( l^{**}_{\alpha}(x) = l^*_{\alpha}(m,r,v) \) for \((m,r,v) \in x\), is a concrete process in \( U(A) \). 

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Due to propositions in section 7 it is straightforward to prove that the correspondence $\alpha \mapsto [L^{**}_\alpha] = [(X^{**}_\alpha, \leq^{**}_\alpha, l^{**}_\alpha)]$ between elements of a clean MTS $A$ and processes in the universe $U(A) = (W(A), V(A), ob(A))$ of objects enjoys the following property.

8.5. Proposition. If $\gamma = \alpha \beta$ with $\text{cod}(\alpha) = \text{dom}(\beta) = c$ then in the category LPOSETS of labelled partially ordered sets and their morphisms the lposet $L^{**}_\gamma$ is the pushout object of the injections of $L^{**}_c$ in $L^{**}_\alpha$ and in $L^{**}_\beta$.

Consequently, we obtain the following result.

8.6. Proposition. Given a clean MTS $A$, the correspondence $\alpha \mapsto [L^{**}_\alpha] = [(X^{**}_\alpha, \leq^{**}_\alpha, l^{**}_\alpha)]$ between elements of $A$ and processes in the universe $U(A) = (W(A), V(A), ob(A))$ of objects is a homomorphism from $A$ to the MTS of processes in $U(A)$.

9 Taking into account unbounded runs

We have defined multiplicative transition system as algebraic structures whose elements, called generalized transitions, may represent runs of a real world system from a state to a state. Now we would like to remark that our definition can easily be generalized such that also unbounded runs, that is runs without initial or final states, can be represented. To this end it suffices to replace the notion of a category by a more adequate notion of a partial category and to modify correspondingly the system of axioms.

A partial category can be defined as a partial algebra $A = (A, \text{dom}, \text{cod}, ;)$ that is defined in exactly the same way as the arrows-only category with the set $A$ of morphisms, the source and the target functions "\text{dom}" and "\text{cod}", and the composition ";"; except that sources and targets may be not defined for some morphisms that are not identities and then the respective compositions are not defined. Limits and colimits in partial categories can be defined as in usual categories.

The system of axioms of multiplicative transition systems can be modified taking into account the concepts and properties of processes and operations on processes described in [Wink 09a] and [Wink 09b]. Namely, it has been shown that the reduct $A = (A, \text{dom}, \text{cod}, ;) = (\text{PROC}(U), \text{dom}, \text{cod}, ;)$ of the algebra $\text{PROC}(U) = (\text{PROC}(U), \text{dom}, \text{cod}, ;, +, 0)$ of processes in a universum $U$ of
objects is a partial category \( \text{pcat}(\text{PROC}(U)) \) such that

(A1') If \( \sigma \alpha \) and \( \sigma' \alpha \) are defined, their targets are defined, and \( \sigma \alpha = \sigma' \alpha \) then \( \sigma = \sigma' \).

(A2') If \( \alpha \tau \) and \( \alpha \tau' \) are defined, their sources are defined, and \( \alpha \tau = \alpha \tau' \) then \( \tau = \tau' \).

(A3') If \( \sigma \tau \) is an identity then \( \sigma \) and \( \tau \) are also identities.

(A4') If \( \sigma \alpha \tau \) is defined, it has a source and a target, and the restriction of the category \( A \) to the set of components of \( \sigma \alpha \tau \) is isomorphic to the restriction of \( A \) to the set of components of \( \alpha \) then \( \sigma \) and \( \tau \) are identities.

(A5') For all \( \xi_1, \xi_2, \eta_1, \eta_2 \) such that \( \xi_1 \xi_2 = \eta_1 \eta_2 \) there exist unique \( \sigma_1, \sigma_2, \eta_1 = \sigma_1 \alpha_2, \eta_2 = \alpha_1 \sigma_2 \).

(A6') If \( (v \xrightarrow{\alpha_1} u \xrightarrow{\alpha_2} w, v \xrightarrow{\alpha_3} w' \xrightarrow{\alpha_4} w) \) is a bicartesian square then for every decomposition \( u \xrightarrow{\alpha_3} v = u \xrightarrow{\alpha_{11}} v_1 \xrightarrow{\alpha_{12}} v \) (resp. \( w \xrightarrow{\alpha_4} u' = w \xrightarrow{\alpha_{11}'} w_1 \xrightarrow{\alpha_{12}'} u' \)) there exist a unique decomposition \( w \xrightarrow{\alpha_1} u' = w \xrightarrow{\alpha_{11}'} w_1 \xrightarrow{\alpha_{12}'} u' \) (resp. \( u \xrightarrow{\alpha_3} v = u \xrightarrow{\alpha_{11}} v_1 \xrightarrow{\alpha_{12}} v \)), and a unique \( v_1 \xrightarrow{\alpha_{12}'} w_1 \), such that \( (v \xrightarrow{\alpha_{11}} u \xrightarrow{\alpha_2} w, v_1 \xrightarrow{\alpha_{12}'} w_1 \xrightarrow{\alpha_{11}'} w) \) and \( (v \xrightarrow{\alpha_{12}} v_1 \xrightarrow{\alpha_{12}'} w_1, v \xrightarrow{\alpha_{12}'} u \xrightarrow{\alpha_{12}'} w_1) \) are bicartesian squares.

(A7') The set \( A \) with morphisms defined as follows is a category \( \text{occ}(A) \):

a morphism from \( \alpha \) to \( \beta \), or an occurrence of \( \alpha \) in \( \beta \), is a triple \( (\sigma, \alpha, \tau) \) such that \( \sigma \alpha \tau = \beta \), or \( \sigma \) and \( \text{dom}(\alpha) \) are not defined and \( \beta = \sigma \alpha \), or \( \tau \) and \( \text{cod}(\alpha) \) are not defined and \( \beta = \sigma \alpha \), or \( \sigma \) and \( \text{dom}(\alpha) \) and \( \text{cod}(\alpha) \) are not defined and \( \beta = \alpha \).

(A8') Every direct system \( D \) in \( \text{occ}(A) \) whose elements are bounded, that is possess sources and targets, has an inductive limit.

(A9') Every \( \alpha \in A \) is the inductive limit of the direct system of its bounded segments, that is of bounded \( \xi \in A \) such that \( \alpha = \alpha_1 \xi \alpha_2 \) for some \( \alpha_1 \) and \( \alpha_2 \).
We remind that in a category a direct system over a directed poset \((I, \leq)\) consists of objects \(a_i\) for \(i \in I\) and of morphisms \(\varphi_{i,j}: a_i \to a_j\) for \(i \leq j\), where \(\varphi_{i,i}\) are identities and \(\varphi_{i,j}\varphi_{j,k} = \varphi_{i,k}\) for \(i \leq j \leq k\), and that the colimit of such a system is called its inductive limit.

As in the case of algebras of bounded processes, the partial category \(\text{pcat}(\text{PROC}(U))\) contains the partial subcategory \(\text{pcatg}(\text{PROC}(U))\) of global processes, and this subcategory enjoys the properties (A1') - (A9'). Consequently, multiplicative transition systems can be defined as partial categories enjoying the properties (A1') - (A9').

It is easy to see that the results obtained for multiplicative transition systems satisfying (A1) - (A6) remain valid for multiplicative transition systems satisfying (A1') - (A9'). However, the problem whether the pomsets corresponding to elements of such systems satisfy (A7'), (A8'), (A9') remains open.

10 Concluding remarks

Making use of the fact that runs of a system from a state to a state and a composition of such runs form a partial algebra satisfying a set of axioms, we have defined a multiplicative transition system, MTS, as an arbitrary partial algebra satisfying this set of axioms, and we have shown that every clean MTS can be viewed as a partial algebra of loose pomsets, or even as a category of processes in a universe of objects. As elements of a clean MTS may represent decomposable runs, algebras of this type become a universal framework for describing systems that may exhibit any combination of discrete and continuous behaviour. As every clean MTS can be viewed as a partial algebra of loose pomsets (or as a category of processes in a universe of objects), such pomsets (processes) become universal basic structures for representing system runs from a state to a state. These results remain true also for MTSs whose elements represent unbounded runs of represented systems.

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