1 Introduction

Theorem 1 (ergodic theorem) Let $(X_i)_{i=-\infty}^{\infty}$ be a stationary ergodic process with $E|X_0| < \infty$. Then
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} X_k = E X_0
\]
holds with probability 1.

Theorem 2 Let $(X_i)_{i=-\infty}^{\infty}$ be a stationary Markov chain, where $P(X_{i+1} = l|X_i = k) = p_{kl}$, $P(X_i = k) = \pi_k$, and the variables take values in a countable set. These conditions are equivalent:

1. Process $(X_i)_{i=-\infty}^{\infty}$ is ergodic.

2. There are no two disjoint closed sets of states; a set $A$ of states is called closed if $\sum_{l \in A} p_{kl} = 1$ for each $k \in A$.

3. For a given transition matrix $(p_{kl})$ there exists a unique stationary distribution $\pi_k$.

Theorem 3 Let $(X_i)_{i=-\infty}^{\infty}$ be a stationary Markov chain with marginal distribution $P(X_i = k) = \pi_k$ and transition probabilities $P(X_{i+1} = l|X_i = k) = p_{kl}$. The entropy rate is
\[
h = -\sum_{kl} \pi_k p_{kl} \log p_{kl}.
\]

2 Task

1. Let us consider a few stationary Markov chains defined by the following transition matrices $P(X_{n+1} = j|X_n = i) = p_{ij}$, where

(a) \[
\begin{pmatrix}
p_{11} & p_{12} \\
p_{21} & p_{22}
\end{pmatrix} = \begin{pmatrix}
1/2 & 1/2 \\
1 & 0
\end{pmatrix};
\]

(b) \[
(p_{ij})_{i,j=1}^{3} = \begin{pmatrix}
1/3 & 1/3 & 1/3 \\
1/4 & 1/4 & 1/2 \\
0 & 3/4 & 1/4
\end{pmatrix};
\]
(c) 

\[(p_{ij})_{i,j=1}^{4} = \begin{pmatrix}
 1/3 & 2/3 & 0 & 0 \\
 3/5 & 2/5 & 0 & 0 \\
 0 & 0 & 1/2 & 1/2 \\
 0 & 0 & 1/4 & 3/4
\end{pmatrix}.\]

2. Tell which of these processes are ergodic and which are nonergodic.

3. For the ergodic process(es) estimate numerically the stationary distribution \(P(X_1 = i) = \pi_i\), using the fact that for almost any choice of \(X_1\) we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} 1\{X_k = i\} = \pi_i.
\]

(To generate values of \(X_k\) use a pseudo-random number generator.) Present the resulted probabilities \(\pi_i\) in a report.

4. Using the estimated probabilities \(\pi_i\) compute the entropy rate \(h\) of the ergodic process(es).

5. Check numerically how the value of limit \(\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} 1\{X_k = i\}\) depends on the initial value of \(X_k\) for the nonergodic process(es).

6. Describe what you have obtained in a report, attach the used scripts, and send it to me (ldebowsk@ipipan.waw.pl).