On processes with summable partial autocorrelations*

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Abstract

A weakly stationary process with summable partial autocorrelations is proved to have one-sided autoregressive and moving average representations. Sums of autocorrelations and alternating autocorrelations are expressed as products of simple rational functions of partial autocorrelations. A general bound for sums of squared autocorrelations in terms of partial autocorrelations is also obtained.

Key words: autocorrelation, partial autocorrelation, autoregressive and moving average representations, Durbin-Levinson algorithm

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1 Introduction

Let $(X_i)_{i\in\mathbb{Z}}$ be a complex-valued weakly stationary process such that $\mathbb{E}X_i = 0$ and $\|X_i\|^2 := \mathbb{E}|X_i|^2 = 1$. Denote by $(\rho_k)_{k\in\mathbb{Z}}$ its autocorrelation function (ACF): $\rho_{i-j} = \text{Cov}(X_i, X_j) = \mathbb{E}(X_i^* X_j)$, where X_i^* is the complex conjugate of X_i. Moreover, $(\alpha_k)_{k\in\mathbb{N}}$ will stand for partial autocorrelation function (PACF) (Durbin, 1960). It is well known that there is a one-to-one correspondence between the ACF and the PACF and, furthermore, (cf. e.g. Ramsey, 1974) parameterization of the process in terms of $(\alpha_k)_{k\in\mathbb{N}}$ is unconstrained, i.e., the sole condition on α_k is $|\alpha_k| \le 1$ and otherwise α_k are free to vary indpendently of one another. Thus it seems natural to investigate classes of processes defined in terms of PACF and to study existing interrelations between asymptotic behavior of (α_k) and (ρ_k) (see e.g. Inoue, 2002). Such study is particularly fruitful in the Gaussian case when ACF and PACF determine the distribution of the process. Then it can be shown (see e.g. Cover and Thomas, 1991, Chapters 2 and 9) that Shannon mutual information between X_1 and X_n equals $I(X_1; X_n) = -\log(1 - |\rho_{n-1}|^2)$ whereas conditional mutual information between X_1 and X_n given $(X_i)_{i=2}^{n-1}$ is $I(X_1; X_n|(X_i)_{i=2}^{n-1}) = -\log(1-|\alpha_{n-1}|^2)$. Hence, (cf. Dębowski, 2005) finiteness of entropy rate or excess entropy, discussed e.g. by Crutchfield and Feldman (2003), can be simply expressed in terms of asymptotics of PACF. Simultaneously excess entropy, defined as mutual information between the past and future of the process, is logarithm of generalized variance (McLeod, 1998) researched in fact already by Grenander and Szegö (1958).

In the present note we study some properties of processes with absolutely summable partial autocorrelations. This condition, close to the requirement of finite excess entropy (cf. Dębowski, 2005), can be viewed as a formal analogue of short-range dependence defined as an absolute summability of ACF. We show in the main result that all such processes have one-sided autoregressive and moving average representations. In particular, it follows that they are purely nondeterministic, i.e., a singular part in the Wold decomposition (Wold, 1938) is null. Moreover, for a real-valued process sums of autocorrelations and alternating autocorrelations turn out to be products of simple rational functions of $(\alpha_k)_{k\in\mathbb{N}}$, namely

$$\sum_{k=-\infty}^{\infty} (\pm 1)^k \rho_k = \prod_{k=1}^{\infty} \frac{1 + (\pm 1)^k \alpha_k}{1 - (\pm 1)^k \alpha_k}. \tag{1}$$

The main result provides also a new bound on partial sums of squared autocorrelations in terms of $(\alpha_k)_{k\in\mathbb{N}}$ which holds for all weakly stationary processes (cf. inequality (19)).

Formula (1) and $|\rho_k| \le 1$ entail the following counterintuitive property:

> If $(\pm 1)^n \alpha_n \ge 0$ for all $n \in \mathbb{N}$ and $(\pm 1)^n \alpha_n \ge a > 0$ holds for at least N distinct n then $(\pm 1)^k \rho_k > 0$ for at least $\left(\frac{1+a}{1-a}\right)^N$ distinct $k \in \mathbb{N}$.

Figure 1 illustrates this property. For a sequence of autoregressive AR(N) such that the pertaining PACF satisfies $\alpha_n^{(N)} \ge a$ for all $1 \le n \le N$, equality (1) implies that $\sum_{k=-\infty}^{\infty} \rho_k^{(N)} \ge ((1 + a)/(1 - a))^N$, which grows exponentially as a function of N. We conjecture that the set of lags for which ACF exceeds a certain positive value grows exponentially with the growing support of PACF.

The paper is organised as follows. In Section 2 we discuss some preliminary facts on linear prediction based on finite and infinite past of the process and state in Theorem 2 a simple result on invertibility. In Section 3 we prove some new results concerning behavior of coefficients of finite predictors. In particular ℓ^1 distance between vectors of predictors' coefficients corresponding to blocks of different length is bounded in terms of PACF. In Section 4 we state and prove the main result.

2 Preliminaries

Let $\Phi_k^{m:n}$ be the best linear predictor of X_k based on of $X_m, X_{m+1}, ..., X_n$ and let ϕ_{nj} satisfy $\Phi_{n+1}^{1:n} = \sum_{j=1}^n \phi_{nj} X_{n+1-j}$. We refer to Pourahmadi (2001, Chapter 7) for a discussion of properties of linear predictors. Partial autocorrelation α_n at lag n is defined as $\mathrm{Corr}(X_{n+1} - \Phi_{n+1}^{2:n}, X_1 - \Phi_1^{2:n})$. Set $\phi_{n0} = -1$ and $\phi_{nj} = 0$ for $j > n$. The Durbin-Levinson (Durbin, 1960) recursion states that

$$\phi_{nj} = \phi_{n-1,j} - \alpha_n^* \phi_{n-1,n-j}^*. \tag{2}$$

Define innovations $Y_p^{m:n} := X_p - \Phi_p^{m:n}$ and $Z_p^n := Y_p^{n+1:p-1} / \left\| Y_p^{n+1:p-1} \right\|$ being standardized innovations pertaining to the projection of X_p on a linear subspace spanned by $\{X_{n+1}, X_{n+2}, \ldots, X_{p-1}\}$. By $\left\|X_i\right\|^2 = 1$ we have $\left\|Y_{n+1}^{1:n}\right\|^2 = \prod_{j=1}^n (1 - |\alpha_j|^2)$ and in view of (2) we obtain

$$Y_p^{p-n:p-1} = -\sum_{k=0}^n \phi_{nk} X_{p-k}$$

and

$$Z_p^{p-n-1} = \sum_{k=0}^n \pi_{nk} X_{p-k}, \tag{3}$$

where $\pi_{nk} = -\phi_{nk} / \prod_{j=1}^n \sqrt{1 - |\alpha_j|^2}$.

Moreover, it is easily seen that we have the following representation of (X_i) in terms of innovations

$$X_p = \sum_{k=0}^n \psi_{nk} Z_{p-k}^{p-n-1}, \tag{4}$$

where

$$\psi_{nk} := \frac{1}{\pi_{n0}} \cdot \begin{cases} 1, & k = 0, \\ -\sum_{j=1}^k \pi_{nj} \psi_{n-j,k-j}, & k \in \{1, ..., n\}. \end{cases} \tag{5}$$

It is an important problem of linear time series to establish conditions under which one is allowed to take $n \to \infty$ in (3) and (4) in order to get AR(∞) and MA(∞) representations of the underlying process. For any weakly stationary process $(X_i)_{i \in \mathbb{Z}}$, we have $\lim_{n\to\infty} \Phi_p^{-n:p-1} = \Phi_p^{-\infty:p-1}$ in $\mathcal{L}^2$, where $\Phi_p^{-\infty:p-1}$ are the best linear predictors of X_p in terms of $(X_n)_{n<p}$ (Pourahmadi, 2001, Section 7.6). If the process is nondeterministic, i.e., $Y_p^{-\infty:p-1} := X_p - \Phi_p^{-\infty:p-1}$ is not equivalent to the null process, then a standard white noise $Z_p := Y_p^{-\infty:p-1} / \left\| Y_p^{-\infty:p-1} \right\|$ is well defined.

Theorem 1 (Pourahmadi, 2001; Dégerine, 1982) *If process $(X_i)_{i\in\mathbb{Z}}$ is nondeterministic then $\lim_{n\to\infty} Z_l^{-n} = Z_l$, $\lim_{n\to\infty}\psi_{nk} = \psi_k$, and $\lim_{n\to\infty}\pi_{nk} = \pi_k$, where $\psi_k := \mathrm{Cov}(Z_{l-k}, X_l)$ and π_k are given by condition*

$$\sum_{j=0}^{k} \pi_j \psi_{k-j} = \begin{cases} 1, & k = 0, \\ 0, & k > 0. \end{cases} \tag{6}$$

Define $\psi_{nk} = \pi_{nk} = 0$ for $k > n$. Process $(X_i)_{i\in\mathbb{Z}}$ is purely nondeterministic, i.e., $X_l = \sum_{k=0}^{\infty} \psi_k Z_{l-k}$, if and only if $\lim_{n\to\infty}\sum_{k=0}^{\infty} |\psi_k - \psi_{nk}|^2 = 0$ (Pourahmadi, 2001, Section 7.6). One can state a similar sufficient condition for $(X_i)_{i\in\mathbb{Z}}$ to have the autoregressive representation, i.e., $Z_l := \lim_{n\to\infty}\sum_{k=0}^{n}\pi_{nk}X_{l-k} = \sum_{k=0}^{\infty}\pi_k X_{l-k}$:

Proposition 1 *Let $(X_i)_{i\in\mathbb{Z}}$ be a weakly stationary process. If $\sum_{k=0}^{\infty}|\pi_{nk}| < \infty$ for all n and (π_k) is such that $\lim_{n\to\infty}\sum_{k=0}^{\infty}|\pi_k - \pi_{nk}| = 0$ then $\sum_{k=0}^{\infty}|\pi_k| < \infty$ and $\sum_{k=0}^{\infty}\pi_k X_{l-k} = \lim_{n\to\infty}\sum_{k=0}^{\infty}\pi_{nk}X_{l-k}$.*

Proof: The convergence follows from the inequality $\|\sum_{k=0}^{\infty}(\pi_{nk} - \pi_k)X_{l-k}\| \le \|X_l\|\sum_{k=0}^{\infty}|\pi_{nk} - \pi_k|$ and completeness of ℓ^1. □

Equality (6) defines π_k given ψ_k and conversely. In fact, (6) is a necessary condition that $Z_l = \sum_{k=0}^{\infty}\pi_k X_{l-k}$ and $X_l = \sum_{k=0}^{\infty}\psi_k Z_{l-k}$ hold simultaneously. Indeed it can be shown that if $X_n = \sum_{k=0}^{\infty}\psi_k Z_{n-k}$ and $\tilde{Z}_n = \sum_{k=0}^{\infty}\pi_k X_{n-k}$ in $\mathcal{L}^2$ for some white noise $(Z_i)_{i\in\mathbb{Z}}$ then coefficients ψ_k and π_k satisfy equality (6) if and only if $\tilde{Z}_n = Z_n$ for all n.

The following result, which seems to have not been stated elsewhere, establishes sufficient conditions under which $X_l = \sum_{k=0}^{\infty}\psi_k Z_{l-k}$ given $Z_l = \sum_{k=0}^{\infty}\pi_k X_{l-k}$, when $(Z_i)_{i\in\mathbb{Z}}$ is not necessarily a white noise process.

Theorem 2 *Assume that ψ_k and π_k satisfy conditions $\sum_{k=0}^{\infty}|\psi_k| < \infty$, $\sum_{k=0}^{\infty}|\pi_k| < \infty$, and (6). Let $(X_i)_{i\in\mathbb{Z}}$ be a weakly stationary process, $Y_n := \sum_{k=0}^{\infty}\pi_k X_{n-k}$, and $\tilde{X}_n := \sum_{k=0}^{\infty}\psi_k Y_{n-k}$. Then $\tilde{X}_n = X_n$.*

Proof: Fix some $M > 0$. By (6) we obtain

$$X_n = \sum_{k=0}^{2M}\sum_{j=0}^{2M}\psi_k\pi_j X_{n-k-j}\mathbf{1}_{\{k+j\le 2M\}},$$

where $\mathbf{1}_{\{\cdot\}}$ denotes the indicator function. Thus for $N > 2M$,

$$X_n - \sum_{k=0}^{M}\psi_k\sum_{j=0}^{N}\pi_j X_{n-k-j} = -\sum_{j=M+1}^{N}\sum_{k=0}^{M}\psi_k\pi_j X_{n-k-j}\mathbf{1}_{\{k+j>2M\}} + \sum_{j=0}^{M}\sum_{k=M+1}^{2M}\psi_k\pi_j X_{n-k-j}\mathbf{1}_{\{k+j\le 2M\}}.$$

Hence

$$\left\|X_n - \sum_{k=0}^{M}\psi_k\sum_{j=0}^{N}\pi_j X_{n-k-j}\right\|^2 \le \left|\sum_{j=0}^{\infty}|\pi_j|\sum_{k=0}^{\infty}|\psi_k| - \sum_{j=0}^{M}|\pi_j|\sum_{k=0}^{M}|\psi_k|\right|^2 \|X_n\|^2.$$

Consequently,

$$\lim_{M\to\infty} \sup_{N>2M} \left\| X_n - \sum_{k=0}^{M} \psi_k \sum_{j=0}^{N} \pi_j X_{n-k-j} \right\| = 0. \tag{7}$$

As $\sum_0^\infty |\pi_k| < \infty$ series $Y_n = \sum_{k=0}^\infty \pi_k X_{n-k}$ converges and $(Y_i)_{i\in\mathbb{Z}}$ is weakly stationary (cf. Brockwell and Davis, 1987, Proposition 3.1.1). Since

$$\left\| \sum_{k=0}^{M} \psi_k \left(Y_{n-k} - \sum_{j=0}^{N} \pi_j X_{n-k-j} \right) \right\| \le \sum_{k=0}^{M} |\psi_k| \left\| Y_n - \sum_{j=0}^{N} \pi_j X_{n-j} \right\|$$

we have

$$\begin{aligned} \left\| X_n - \tilde{X}_n \right\| \le & \left\| X_n - \sum_{k=0}^{M} \psi_k \sum_{j=0}^{N} \pi_j X_{n-k-j} \right\| + \left\| \tilde{X}_n - \sum_{k=0}^{M} \psi_k Y_{n-k} \right\| \\ & + \sum_{k=0}^{M} |\psi_k| \left\| Y_n - \sum_{j=0}^{N} \pi_j X_{n-j} \right\|. \end{aligned}$$

Taking $\lim_{M\to\infty} \sup_{N>2M}$ of the right hand side, in view of (7) we obtain $\left\| X_n - \tilde{X}_n \right\| = 0$. □

To relate the asymptotics of ψ_k and π_k, we need the following extension of Theorem 3.1.1 in Brockwell and Davis (1987):

Proposition 2 *For ψ_k and π_k satisfying (6), define $\psi(z) := \sum_{k=0}^\infty \psi_k z^k$ and $\pi(z) := \sum_{k=0}^\infty \pi_k z^k$. Assume that $\pi(z)$ converges for $|z| \le 1$ and $\pi(z) \ne 0$ for all $|z| \le 1$.*

(i) If $\pi(z)$ converges for $|z| < r$ where $r > 1$ then $\sum_{k=0}^\infty |\psi_k| < \infty$ and $\psi(z) = 1/\pi(z)$ for $|z| < r''$ with some $r'' \in (1, r)$.

(ii) If $\pi(z)$ is continuous for $|z| \le 1$ then $1/\pi(z)$ is continuous for $|z| \le 1$, $\sum_{k=0}^\infty |\psi_k|^2 < \infty$, and $\psi(z) = 1/\pi(z)$ for all z with $|z| < 1$ and almost all z with $|z| = 1$.

Proof:

(i) We choose $a > 0$ and $r'' \in (1, r)$ such that $|\pi(z)| > a$ for $|z| < r''$. Hence $1/\pi(z)$ is analytic for $|z| < r''$. By Abel's theorem, $1/\pi(z) = \psi(z)$. Series $\psi(z)$ converges for $|z| < r''$ and thus $\sum_{k=0}^\infty |\psi_k| < \infty$.

(ii) We have $\psi(z) = 1/\pi(z)$ for $|z| < 1$ since $\pi(z)$ is analytic there. Since $|\pi(z)| > a$ for $|z| \le 1$ and some positive a, $1/\pi(z)$ is continuous and $\int_{-\pi}^{\pi} |\psi(re^{i\omega})|^2 d\omega < 2\pi a^{-2}$ for all $r < 1$. From this it follows that $\sum_{k=0}^\infty |\psi_k|^2 < \infty$ and $\lim_{r\to 1_-} \psi(re^{i\omega}) = \psi(e^{i\omega}) = \lim_{r\to 1_-} 1/\pi(re^{i\omega}) = 1/\pi(e^{i\omega})$ for almost all ω (Grenander and Szegö, 1958, Sections 1.1, 1.13).

□

Proposition 3 *For a standard nondeterministic process* $(X_i)_{i\in\mathbb{Z}}$*, let process* $(Z_i)_{i\in\mathbb{Z}}$ *and coefficients* ψ_k *be those appearing in Theorem 1. The following conditions are equivalent:*

(i) $X_l = \sum_{k=0}^{\infty} \psi_k Z_{l-k}$*,*

(ii) $\rho_k = \sum_{j=0}^{\infty} \psi_{j+k}^* \psi_j$ *for* $k \geq 0$*,*

(iii) $\rho_k = (2\pi)^{-1} \int_{-\pi}^{\pi} |\psi(e^{i\omega})|^2 e^{ik\omega} d\omega$*.*

Proof: Condition (i) implies (ii) even when $\sum_{j=0}^{\infty} |\psi_j| = \infty$ (for summable ψ_k see Brockwell and Davis, 1987, Proposition 3.1.2). To modify the proof by Brockwell and Davis, notice that $X_l = \sum_{k=0}^{N} \psi_k Z_{l-k} + \sum_{k=N+1}^{\infty} \psi_k Z_{l-k}$, whence we obtain $\left|\rho_k - \sum_{j=0}^{N} \psi_{j+k}^* \psi_j\right|^2 \leq \sum_{j=N+1}^{\infty} |\psi_j|^2$ via Schwarz inequality. Under square summability of ψ_k, Grenander and Szegö (1958, Sections 1.13, 1.14) proved that (ii) implies (iii). Finally, observe the following. If $X_l = \sum_{k=0}^{\infty} \psi_k Z_{l-k} + V_l$ where $\|V_l\| > 0$ then $(Z_i)_{i\in\mathbb{Z}}$ and $(V_i)_{i\in\mathbb{Z}}$ are uncorrelated by Wold decomposition. It follows that $\rho_0 = \|X_l\|^2 > \sum_{k=0}^{\infty} |\psi_k|^2 = (2\pi)^{-1} \int_{-\pi}^{\pi} |\psi(e^{i\omega})|^2 d\omega$. Hence, (iii) implies (i). □

By the Durbin-Levinson recursion, Theorem 1, and Proposition 3, for purely nondeterministic processes, sequences $(\alpha_k)_{k\in\mathbb{N}}$, $(\pi_k)_{k\in\mathbb{N}}$, $(\psi_k)_{k\in\mathbb{N}}$, and $(\rho_k)_{k\in\mathbb{N}}$ determine each another.

3 Auxiliary results on finite predictiors

This section contains new results on coefficients ϕ_{nj}. They are used to prove the main result but may be interesting in their own right. No conditions are imposed besides weak stationarity of the process. In the first proposition we upper-bound $|\phi_{nj}|$ by the absolute values of the coefficients obtained when α_n in the Durbin-Levinson recursion is replaced by $-|\alpha_n|$.

Proposition 4 *Define* ϕ_{nj}^A *by substituting* ϕ_{nj}^A *for* ϕ_{nj} *and* $-|\alpha_n|$ *for* α_n *in formula (2). Then for* $n \in \mathbb{N}$ *and* $j \in \{0, 1, \ldots, n\}$

$$\phi_{nj}^A \leq 0, \tag{8}$$

$$|\phi_{n+1,j}^A| \geq |\phi_{n,j}^A|, \tag{9}$$

$$\sum_{j=0}^{n} |\phi_{nj}^A| = \prod_{k=1}^{n} (1 + |\alpha_k|), \tag{10}$$

$$|\phi_{nj}| \leq |\phi_{nj}^A|, \tag{11}$$

where the product over empty set of indices is defined as 1.

Proof: We proceed by induction on n. As $\phi_{10}^A = \phi_{00}^A = \phi_{00} = -1$ it is immediate that (8)–(11) hold for $n = 0$. Assume that (8)–(11) are valid for $n = m - 1$. In view of (2) $\phi_{mj}^A = \phi_{m-1,j}^A + |\alpha_m| \phi_{m-1,m-j}^A$ for $j < m$ and $\phi_{mm}^A = -|\alpha_m|$. As $\phi_{m-1,j}^A, \phi_{m-1,m-j}^A \leq 0$ by (8) for $n = m - 1$, (8) and (9) follow for $n = m$. Equality (10) follows from $\sum_{j=0}^{m} \phi_{mj}^A = (1 + |\alpha_m|) \sum_{j=0}^{m-1} \phi_{m-1,j}^A$ and the inductive assumption. Finally, if (11) holds for $n = m - 1$ then $|\phi_{mj}| \leq |\phi_{m-1,j}| + |\alpha_m||\phi_{m-1,m-j}| \leq |\phi_{m-1,j}^A| + |\alpha_m||\phi_{m-1,m-j}^A| = |\phi_{mj}^A|$. □

Note that in view of (10) sums $\sum_{j=0}^{n} |\phi_{nj}|$ grow exponentially for some nondeterministic processes, e.g., in the case when $|\alpha_k| \geq a$ for all k and some positive a. However, if $\prod_{k=1}^{\infty} (1 + |\alpha_k|) < \infty$ then $(\phi_{n\cdot})_{n\in\mathbb{N}}$ converges in ℓ^1.

Theorem 3 *Let $\phi_{nj} := 0$ for $j > n$. If $m > n$ then*

$$\sum_{j=0}^{m} |\phi_{mj} - \phi_{nj}| \leq \prod_{k=1}^{m} (1 + |\alpha_k|) - \prod_{k=1}^{n} (1 + |\alpha_k|) \,. \tag{12}$$

Proof: Recursion (2) together with (10) and (11) imply

$$\begin{aligned} \sum_{j=0}^{m} |\phi_{mj} - \phi_{m-1,j}| &\leq |\alpha_m| \sum_{j=0}^{m-1} |\phi_{m-1,j}| \leq |\alpha_m| \prod_{k=1}^{m-1} (1 + |\alpha_k|) \\ &= \prod_{k=1}^{m} (1 + |\alpha_k|) - \prod_{k=1}^{m-1} (1 + |\alpha_k|) \,. \end{aligned}$$

Using repeatedly the triangle inequality we obtain (12). □

In contrast to a similar Baxter inequality (Baxter, 1962) which stipulates some conditions on spectral density, relation (12) holds for all weakly stationary processes.

Define polynomials $\pi_n(z) := \sum_{k=0}^{n} \pi_{nk} z^k$ and $\bar{\pi}_n(z) = z^n [\pi_n(1/z^*)]^*$, $z \in \mathbb{C}$ (Grenander and Szegö, 1958). Recursion (2) is equivalent to $\pi_0(z) = \bar{\pi}_0(z) = 1$ and

$$\begin{bmatrix} \pi_n(z) \\ \bar{\pi}_n(z) \end{bmatrix} = L(\alpha_n) \begin{bmatrix} \pi_{n-1}(z) \\ z\bar{\pi}_{n-1}(z) \end{bmatrix}, \qquad L(u) = \frac{1}{\sqrt{1 - |u|^2}} \begin{bmatrix} 1 & -u^* \\ -u & 1 \end{bmatrix}.$$

Theorem 4 *We have*

$$|\pi_n(z)|^2 - |\bar{\pi}_n(z)|^2 = |\pi_{n-1}(z)|^2 - |z|^2 |\bar{\pi}_{n-1}(z)|^2, \tag{13}$$

$$|\pi_n(z)|^2 = |\bar{\pi}_n(z)|^2 + (1 - |z|^2) \sum_{k=0}^{n-1} |\bar{\pi}_k(z)|^2 \geq 1 - |z|^2. \tag{14}$$

Proof: Let $[a, b]^T := L(u)[c, d]^T$ with x^T denoting the transpose of x and observe that $|c|^2 - |d|^2$ is an invariant of Lorentz transformation $L(u)$: $|a|^2 - |b|^2 = |c|^2 - |d|^2$. Hence (13) follows. In order to prove (14) first note that $\pi_0(z) = \bar{\pi}_0(z) = 1$. Writing (13) for $n = k$: $|\pi_k(z)|^2 - |\pi_{k-1}(z)|^2 = |\bar{\pi}_k(z)|^2 - |z|^2 |\bar{\pi}_{k-1}(z)|^2$ and summing the equality from $k = 1$ to $k = n$, we obtain (14). □

Inequality (14) implies that polynomials $\pi_n(z)$ and $\phi_n(z) := \sum_{k=0}^{n} \phi_{nk} z^k = -\pi_n(z)/\pi_{n0}$ do not have zeros for $|z| < 1$. A more involved proof of the latter fact, making use of spectral properties of $(X_i)_{i\in\mathbb{Z}}$ was given by Grenander and Szegö (1958, Section 2.3(a)). The idea of using the invariant of Lorentz transform seems to be new.

Theorem 5 *We have*

$$\phi_n(\pm 1) = \prod_{k=1}^{n} \left(1 - (\pm 1)^k \alpha_k\right) \quad \textit{if } \alpha_k \in \mathbb{R} \textit{ for } k \leq n; \tag{15}$$

$$|\phi_n(z)| \in \left[\prod_{k=1}^{n} (1 - |\alpha_k|) \,, \prod_{k=1}^{n} (1 + |\alpha_k|)\right] \quad \textit{for } |z| \leq 1. \tag{16}$$

Proof: If α_k is real for $k \leq n$ then $\phi_n(\pm 1) = (1 - (\pm 1)^n \alpha_n)\, \phi_{n-1}(\pm 1)$. Hence we have (15).

Observe that for $|z| = 1$ and $A \in \mathbb{C}$ we have

$$(1 - |\alpha_n|) \cdot |A| \leq \left| A - [\alpha_n z^{-n} A]^* \right| \leq (1 + |\alpha_n|) \cdot |A|. \tag{17}$$

Formula (17) implies (16) for $|z| = 1$ since $\phi_n(z) = \phi_{n-1}(z) - [\alpha_n z^{-n} \phi_{n-1}(z)]^*$. By the maximum principle for $\phi_n(z)$ (Rudin, 1974, Section 12.1), we also have $|\phi_n(z)| \leq \prod_{k=1}^n (1 + |\alpha_k|)$ for $|z| \leq 1$. If $\prod_{k=1}^n (1 - |\alpha_k|) = 0$ then the second inequality in (16) is trivial. Otherwise, by Theorem 4, $\phi_n(z) \neq 0$ holds not only for $|z| = 1$ but also for $|z| < 1$. Thus we obtain $|\phi_n(z)| \geq \prod_{k=1}^n (1 - |\alpha_k|)$ for $|z| \leq 1$ using the maximum principle for $1/\phi_n(z)$. □

4 The main result

Theorem 6 *Let $(X_i)_{i \in \mathbb{Z}}$ be a standard weakly stationary process such that $\sum_{k=1}^\infty |\alpha_k| < \infty$ and $|\alpha_k| < 1$ for all k. Then:*

(i) $Z_l = \sum_{k=0}^\infty \pi_k X_{l-k}$ and $X_l = \sum_{k=0}^\infty \psi_k Z_{l-k}$ for $(Z_i)_{i \in \mathbb{Z}}$ being the standard white noise pertaining to Wold decomposition.

(ii) For $|z| \leq 1$ series $\pi(z) = \sum_{k=0}^\infty \pi_k z^k$ satisfies

$$\sum_{k=0}^\infty |\pi_k|, |\pi(z)| \in \left[\prod_{k=1}^\infty \sqrt{\frac{1 - |\alpha_k|}{1 + |\alpha_k|}}, \prod_{k=1}^\infty \sqrt{\frac{1 + |\alpha_k|}{1 - |\alpha_k|}} \right]. \tag{18}$$

(iii) Series $\psi(z) = \sum_{k=0}^\infty \psi_k z^k$ satisfies $\psi(z) = 1/\pi(z)$ for all z with $|z| < 1$ and almost all z with $|z| = 1$.

(iv) For $n \in \mathbb{N}$ we have

$$\sum_{k=-n}^n |\rho_k|^2 \leq \prod_{k=1}^n \left(\frac{1 + |\alpha_k|}{1 - |\alpha_k|} \right)^2. \tag{19}$$

(v) If $\alpha_k \in \mathbb{R}$ for all k then (1) holds and moreover

$$\pi(\pm 1) = \prod_{k=1}^\infty \sqrt{\frac{1 - (\pm 1)^k \alpha_k}{1 + (\pm 1)^k \alpha_k}}. \tag{20}$$

The left hand side of (1) stems from the moving average representation while the right hand side pertains to the autoregressive representation. Note that $\sum_{k=0}^\infty (\pm 1)^k \rho_k$ appearing in (1) equals the value of the spectral density at 0 or π depending on the considered sign of ± 1. Since ρ_n considered as a function of $(\alpha_k)_{k \in \mathbb{N}}$ is a polynomial of $\alpha_1, ..., \alpha_n$, inequality (19) holds in fact for *all* weakly-stationary processes. Nevertheless, it is an open problem to extend (1) to some classes of processes with absolutely insummable PACF.

Proof of Theorem 6: It is convenient to split the proof of (v) into two parts appended to the proofs of other parts of the theorem. Namely, we prove (20) while proving (ii) and we justify (1) when proving (iv). We also insert the proof of the second part of (i) into the proof of (iv).

(i) Since $\sum_{k=1}^{\infty} |\alpha_k| < \infty$, we have $\prod_{k=1}^{\infty}(1+|\alpha_k|) < \infty$. By Theorem 3, $(\phi_{n\cdot})_{n\in\mathbb{N}}$ is a Cauchy sequence in ℓ^1. Thus limits $\phi_k := \lim_{n\to\infty} \phi_{nk}$ exist and satisfy $\lim_{n\to\infty} \sum_{k=0}^{\infty} |\phi_{nk} - \phi_k| = 0$. We have $\pi_n(z) = -\pi_{n0}\phi_n(z)$ and $\lim_{n\to\infty} \pi_{n0} = \pi_0 = \prod_{j=1}^{\infty}[1-|\alpha_j|^2]^{-1/2} \in (0,\infty)$. Hence the process is nondeterministic and $(\pi_{n\cdot})_{n\in\mathbb{N}}$ is also a Cauchy sequence in ℓ^1. By Proposition 1 and Theorem 1, $Z_l = \sum_{k=0}^{\infty} \pi_k X_{l-k}$ and thus the first part of (i) is proved.

(ii) Since $(\phi_{n\cdot})_{n\in\mathbb{N}}$ is a Cauchy sequence in ℓ^1, $\phi_n(z)$ converge uniformly to $\phi(z)$ for $|z| \le 1$, where $\phi(z) := \sum_{k=0}^{\infty} \phi_k z^k$. Therefore, properties (16) and (15) apply to $\phi(z)$ if we substitute $\phi(z)$ for $\phi_n(z)$ and $\prod_{k=1}^{\infty}$ for $\prod_{k=1}^{n}$. Since $\pi(z) = -\pi_0\phi(z)$ then (18) and (20) for $\pi(z)$ follow from (16) and (15) respectively. Formula (18) for $\sum_{k=0}^{\infty} |\pi_k|$ follows from (10)–(11), $\sum_{k=0}^{\infty} |\pi_k| \ge |\pi(0)|$, and (18) for $\pi(z)$.

(iii) Because $\pi_n(z)$ converge uniformly to $\pi(z)$, $\pi(z)$ is continuous for $|z| \le 1$. By formula (18), there exists $a > 0$ such that $|\pi(z)| > a$ for all $|z| \le 1$ so we obtain (iii) from Proposition 2.(ii).

(iv) Observe first that if $(X_i^{(n)})_{i\in\mathbb{Z}}$ is an autoregressive process AR(n) with $\pi_n^{(n)}(z) = \pi_n(z)$ then $\pi^{(n)}(z) = \pi_n^{(n)}(z)$ converges for $|z| < \infty$. By Proposition 2.(i), $\sum_{k=0}^{\infty} |\psi_k^{(n)}| < \infty$ so, by Theorem 2, $X_l^{(n)} = \sum_{k=0}^{\infty} \psi_k^{(n)} Z_{l-k}^{(n)}$ (this also follows from Theorem 3.1.1 by Brockwell and Davis, 1987).

In order to use this observation, note that (3) implies that the spectral density of $(X_i^{(n)})_{i\in\mathbb{Z}}$ equals $|1/\pi_n(e^{i\omega})|^2$ and its ACF and PACF values coincide with ρ_k and α_k for $0 \le k \le n$. By Proposition 3 applied to $(X_i^{(n)})_{i\in\mathbb{Z}}$,

$$\rho_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} |1/\pi_n(e^{i\omega})|^2 e^{ik\omega} d\omega \quad \text{for } 0 \le k \le n. \tag{21}$$

Functions $1/\pi_n(z)$ converge uniformly to $1/\pi(z)$ for $|z| \le 1$ since $|\pi(z)| > a$ for $|z| \le 1$. Hence (21) implies $\rho_k = (2\pi)^{-1}\int_{-\pi}^{\pi} |\psi(e^{i\omega})|^2 e^{ik\omega} d\omega$ for all $k \ge 0$. By Proposition 3, $X_l = \sum_{k=0}^{\infty} \psi_k Z_{l-k}$ and thus the second part of (i) is proved.

By the Riesz-Fischer theorem and the Parseval identity, we have $\sum_{k=-\infty}^{\infty} |\rho_k|^2 = (2\pi)^{-1}\int_{-\pi}^{\pi} |\psi(e^{i\omega})|^4 d\omega$ and $\sum_{k=-\infty}^{\infty} \rho_k e^{-ik\omega} = |\psi(e^{i\omega})|^2$. Thus formulae (20) (proved above) and (18) imply (1) and

$$\sum_{k=-\infty}^{\infty} |\rho_k|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\pi(e^{i\omega})|^{-4} d\omega \le \prod_{k=1}^{\infty}\left(\frac{1+|\alpha_k|}{1-|\alpha_k|}\right)^2. \tag{22}$$

Applying (22) to $(X_i^{(n)})_{i\in\mathbb{Z}}$ defined above and noting that its PACF equals 0 for lags greater than n we get (19).

□

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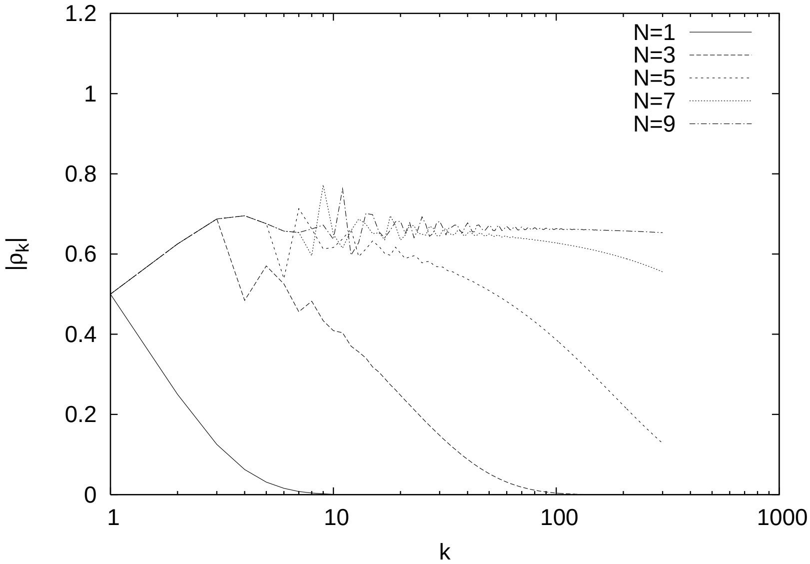

Figure 1: The plot of $|\rho_k|$ for $\alpha_n = 0.5$ if $n \leq N$ and $\alpha_n = 0$ otherwise.