Pure equilibria in a simple dynamic model of strategic market game

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1 Introduction

Consider the game as follows: Two players hold integral amounts of money with total amount equal to M. Every day one unit of some nondurable commodity is brought to market and players bid some integral parts of their money for the good. Portions of the good are awarded to players according to their bids and on the same day they consume it. On the next day the money paid by the players is redistributed — each player receives a random share according to some probability distribution. Then another unit of the commodity is brought to market and bids are made, and so on.

Different rules of distribution of the good based on their bids, as well as different rules of redistribution of the money in the game are possible. Also players may have different utility functions and different discount factors. So even in this simple model of the game there is a lot of place for research.

A discrete model of two-person stochastic game of this type was presented by Secchi and Sudderth [?]. Two mechanisms of distribution of the good after the bids are made were considered. One was winner-takes-all rule, which prescribed to sell whole of the good to the agent who makes the largest bid. The second rule divided rewards to the players proportionally to their bids. In both situations the results about the form of optimal pure strategies in the game, when discount factor is small or approaches 1, were obtained. However, no genaral results on the existence of pure-strategy stationary equilibria were shown.

In our paper we consider a fixed-sum model similar to that of [?]. For this model we prove that the stochastic game possesses a pure-strategy equilibrium for all discount factors, and discuss some general features of optimal strategies. In addition we show that results of Secchi and Sudderth about the form of optimal strategies for small values of β and β approaching 1 still hold for our model.

2 Model and main results

The game described in first section is a constant-sum stochastic game and can be described with use of the following four items:

1. The state of the game in some moment of time is the amount of money kept by Player 1 at this moment. Thereby state space for the game is the set $S = \{0, 1, \ldots, M\}$, where M is fixed integral amount of money held by the two players.

2. Actions of the players represent their bids, and thus, the sets of actions available to player 1 and player 2 in state x of the game are $\{0, 1, \ldots, x\}$ and $\{0, 1, \ldots, M - x\}$ respectively.

3. Daily reward to player 1 in state x, when players use actions a and b respectively is
$$r(x, a, b) = \begin{cases} 1 & \text{when } a > b, \\ 1/2 & \text{when } a = b, \\ 0 & \text{when } a < b. \end{cases}$$
Daily reward of player 2 is equal to $\tilde{r} = 1 - r(x, a, b)$.

4. The law of motion between states q is defined for every triple (x, a, b) as follows:
$$q(x, a, b) = \begin{cases} \frac{1}{2}\delta[x] + \frac{1}{2}\delta[x + b] & \text{when } a < b, \\ \frac{1}{4}\delta[x - a] + \frac{1}{2}\delta[x] + \frac{1}{4}\delta[x + a] & \text{when } a = b, \\ \frac{1}{2}\delta[x] + \frac{1}{2}\delta[x - a] & \text{when } a > b, \end{cases}$$
where $\delta[y]$ denotes a degenarate probability distribution concentrated in point y.

   q defined as above reflects the fact that on every stage of the game two decisions can be made using "fair coin toss". One is done in every case and it decides what will be the distribution of the money paid by the players for the portion of good in the next stage of the game. The other decision is made only when bids of both players are equal. Then only one of them is chosen to consume the good and pay for it with a help of coin toss.

Two main results of the paper are the following:

Theorem 2.1 *The game Γ_β has a value $V_\beta(x)$ satisfying the following:*

1. *For every $\beta \in (0,1)$ it is nondecreasing in state $x \in S$ and for every $x < M$,*
$$V_\beta(x+1) - V_\beta(x) \leq 1. \tag{1}$$

2. *For every fixed $x \in S$ it is nondecreasing in β.*

3. *For every $x \in S$ the following equality holds:*
$$V_\beta(x) + V_\beta(M-x) = \frac{1}{1-\beta}.$$

Theorem 2.2 *For every $\beta \in (0,1)$ the game Γ_β possesses a symmetric pure strategy stationary equilibrium (f_β^1, f_β^2) (i.e. for every $x \in S$, $f_\beta^1(x) = f_\beta^2(M-x)$) such that*

1. *For every x, $|f_\beta^1(x) - f_\beta^2(x)| \leq 1$.*

2. *$f_\beta^1(0) = 0$ and for $x > 0$, $f_\beta^1(x) > 0$.*

3. *For every $x < M$,*
$$f_\beta^2(x) - 1 \leq \min\{f_\beta^1(x+1), f_\beta^2(x+1)\} \leq \begin{cases} f_\beta^2(x) + 1, & \text{if } f_\beta^1(x) = f_\beta^2(x) \\ f_\beta^2, & \text{if } f_\beta^1(x) \neq f_\beta^2(x) \end{cases}$$

The result obtained in the above theorem seems very technical, so it needs a bit of explanation. The first two parts are obvious. One states that it is never optimal for the players to waste money on bids that would be much larger then the bids of the other player. The second one states that it is always more profitable to play than to let the other player decide everything on his own. The third part is much more complicated. It is easy to notice for a careful reader that it implies that optimal strategies of the players satisfy

Lipschitz property with constatnt 2. On the other hand, by writing it in more complicated way we wanted to emphasize that the class of optimal solutions in this game has a lot more restrictive structure, which ties in a sense the forms of the strategies of the two players.

The third main result is the version of Theorems 3.1 and 4.2 in Secchi and Sudderth [**?**] for our model. To formulate it we need to define two specific types of strategies used in those theorems. Namely, strategy

$$\sigma_1(x) = \begin{cases} x & \text{if } x \leq \frac{M}{2} \\ M - x + 1 & \text{if } x > \frac{M}{2} \end{cases}$$

of player 1 (and similarly strategy $\sigma_2(x) = \sigma_1(M - x)$ of player 2) is called *bold*.

Strategy

$$\tau_1(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 1 \end{cases}$$

of player 1 (and similarly strategy $\tau_2(x) = \tau_1(M - x)$ of player 2) is called *timid*.

Theorem 2.3 *1. For every $\beta \leq 2 - \sqrt{2}$, the pair (σ_1, σ_2) is an equilibrium in the game.*

2. There exists $\beta_0 < 1$ such that for every $\beta \in (\beta_0, 1)$ (τ_1, τ_2) is an equilibrium in the game.

References

[1] Secchi, P., Sudderth, W.D. (2003) A Simple Two-Person Stochastic Game with Money. In *Annals of Dynamic Games* vol.7 (Nowak Ed.), Birkhäuser, Boston (to appear)