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A Necessary and Sufficient Condition for a Homogenous Generalized Assignment Game to Have a Nonempty Core.

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Summary

A generalized assignment game is a special *n*-person game in characteristic function form, associated with an *exchange network*. The latter term is to denote any abstract socio-economic system made up of:

(1) a *transaction opportunity graph* G=(N,L) which is a connected undirected graph whose node set N represents the positions occupied by *n actors* who are given as a group an opportunity to negotiate and conclude bilateral transactions with the restriction that a deal between the actors in positions P and Q can be made only if line PQ is in the line set L of G;

(2) a *profit pool network* C over G, or a mapping which assigns to any line $PQ \in L$ a number $C_{PQ}>0$ interpreted as the size of a *profit pool* to be divided between the occupants of positions P and Q; the term *transaction* is referred to any pool split $x_{PQ}+x_{QP}=C_{PQ}$ agreed-on by two bargainers;

(3) an *exchange regime* defined as a family **T** of subsets of L called *transaction sets*.

It assumed that for any line $PQ \in L$ there exists a transaction set $T \in \mathbf{T}$ such $PQ \in T$. Transaction sets represent all configurations of bilateral agreements which may happen in a single negotiation round in accordance with certain rules which impose certain limitations on the locus and number of transactions across the network.

A transaction set T is called *maximal* is there is no T' in **T** such that $T \subset T'$ and $T \neq T'$. That is, if for any line PQ in T the bargainers in positions P and Q have come to terms, no further transactions can be concluded, the negotiation round comes to an end, and P and Q get their negotiated shares x_{PQ} and x_{QP} of C_{PQ}.

A *cumulative* exchange regime is defined by the following condition: for any $T \in \mathbf{T}$ and for any $T' \subset T$, $T' \in \mathbf{T}$. Under a cumulative exchange regime, any set {PQ} is in **T** so that any two actors who have settled on a pool split can safely wait for the end of a round because their payoffs do not depend on what transaction are later concluded elsewhere in the network.

An exchange regime is called *additive* if $T \cup T' \in \mathbf{T}$ for any two point-disjoint transaction sets $T, T' \in \mathbf{T}$. T and T' are *point-disjoint* if there is no point P such that $PQ \in T$ and $PQ' \in T'$ for some points Q and Q'. To give an example of a cumulative and additive exchange regime, consider the *k-exchange rule* which permits to every actor to make at most k transaction per round. Let $\mathbf{T_1}(G)$ denote the *one-exchange regime* generated by 1-exchange rule. Then $T \in \mathbf{T_1}(G)$ if and only if no two lines in T have a common endpoint.

In graph theory (see Harary 1969, chapter 10) any subset T of L with this property is called a *matching* or an *independent set of lines*. The structural parameter of $G=(N,L)$ known as the *line independence number* is defined as $\beta_1(G)=Max\{|T|:T \in \mathbf{T_1}(G)\}$ where $|T|$ stands for the cardinality of T. A matching T is called *optimal* if $|T|=\beta_1(G)$ (the term "maximal" in the meaning "of maximum cardinality" usually found in graph theory literature is replaced here with "optimal" to avoid confusion with "maximality with respect to inclusion").

The *n*-person game associated with an exchange network C over $G=(N,L)$ with a cumulative and additive exchange regime $\mathbf{T}$ is defined as $(N, v_{C,\mathbf{T}})$ where N, or the set of network nodes, becomes the *set of players*, and the *value of the characteristic function* $v_{C,\mathbf{T}}$ for any *coalition* $S \subset N$ is computed according to the formula

$$v_{C,\mathbf{T}}(S) = \underset{T}{Max} \sum_{PQ \in T} C_{PQ}$$

in which the maximum is taken over all transaction sets T such that both endpoints P and Q of any line $PQ \in T$ lie in S. Note that $v_{C,\mathbf{T}}(\{P\})=0$ and $v_{C,\mathbf{T}}(\{P,Q\})=C_{PQ}$ for any $P,Q \in N$. The values assumed by $v_{C,\mathbf{T}}$ for triplets and larger coalitions depend on a concrete exchange regime. The properties of cumulativity and additivity imply that the game so defined is *superadditive* and *essential*.

The theory of characteristic function games associated with general exchange networks has yet to be developed save for the case of one-exchange regime which is covered extensively in chapter 5 of the author's monograph (*The Mathematics of Exchange Networks*).

The game (N,v_C) where $v_C=v_{C,\mathbf{T1}(G)}$ was first defined by Bienenstock and Bonacich (Bienenstock and Bonacich 1992, Bonacich and Bienenstock 1993). Games over bipartite graphs (G is *bipartite* if N is a union of two disjoint subsets N_1 and N_2 such that any line in L has one endpoint in N_1 and the other in N_2) with one-exchange rule, called *two-sided assignment games*, had been studied in mathematical economics since the time when Shapley and Shubik (1972; see also Shubik 1984) demonstrated that every game of the kind has a nonempty core. As regards *generalized assignment games* as we propose to name the games associated with one-exchange networks over arbitrary graphs, a necessary and sufficient condition of core nonemptiness has yet to be found.

This paper offers a solution to the problem for a special class of one-exchange networks, the *homogenous* networks in which every line in L has been assigned a profit pool of the same size C. As a consequence, whether the game associated with such a network does or does not have a nonempty core depends solely on the structure of the underlying graph. One can assume without a loss of generality that C=1. Let v_G denote the characteristic function of the *homogenous generalized assignment game over G*. It's not difficult to verify that $v_G(S)=\beta_1(G_S)$ where $G_S=(S,L_S)$ is the *subgraph of* $G=(N,L)$ *generated by S* ($PQ \in L_S$ if and only if $PQ \in L$ and $P,Q \in S$). Clearly, $v_G(N)=\beta_1(G)$.

We show in this paper that the existence of the core for the game (N,v_G) depends on the relation between $\beta_1(G)$ and the *point covering number* $\alpha_0(G)$ defined (see Harary

1969: chapter 10) as the smallest number of nodes which *cover all lines of G*. More exactly, a subset S of the node set N of G is called a *point cover* for G if every line of G has at least one endpoint in S. Then, $\alpha_0(G)=Min\{|S|\}$ with the minimum taken over all point covers for G.

Following Bonacich (1998, 1999) we consider for every connected graph $G=(N,L)$ its subgraph $G^o=(N,L^o)$ whose line set L^o consists of all optimal lines in L; a *line* $PQ\in L$ is called *optimal* if $PQ\in T$ for some optimal matching T. If G^o is connected, then G and the respective homogenous one-exchange network is said to be *game-indecomposable*. If G^o is not connected, then there exists a partition of N into pairwise disjoint nonempty subsets $N_1,\ldots.N_m$ such that all the subgraphs $G^o{}_i=G^o{}_{Ni}$ of G^o generated by the N_i are connected. The subgraphs $G_i=G_{Ni}$ of G generated by the same subsets of N are called *game-components* of G.

The main results to be presented in this paper are the following two theorems.

THEOREM 1. *If a homogenous one-exchange network over G is game-indecomposable, then the core of the generalized assignment game* (N,v_G) *is not empty if and only if* $\beta_1(G)=½n$ *or* $\beta_1(G)=\alpha_0(G)$.

THEOREM 2. *The homogenous generalized assignment game* (N,v_G) *has a nonempty core if and only the games* (N_i,v_{Gi}) *associated with all game-components* $G_i=(N_i,L_i)$ *of* $G=(N,L)$ *have nonempty cores*.

The demonstrations of these theorems are going to be presented in public for the first time at this conference.

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