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1. Introduction *Games with constraints* were firstly considered by Debreu and Arrow. Their main result generalizes Glicksberg's Theorem about the existence of pure Nash equilibria in n-person games. Debreu considered the case of n-person games in which the players are restricted in choosing their strategies in the sense that the "effective" set of pure strategies of each player can depend on decisions taken by the remaing players. Thus in such a model of a game, some multistrategies (vectors describing the players' choices) are not admissible, and we should look for an equilibrium only among admissible ones. Games with constraints were often used as a useful tool in searching for equilibria in models of economic games.

The second approach to games with constraints is that instead of a game with a set of "forbidden" multistrategies (not admissible), a generalized model of classical non-cooperative games is considered where some multistrategies determine players' payoffs equal to $-\infty$. As hitherto, games with such infinite payoffs have not been considered in term of the existence of finite Nash equilibria. One of our results in the paper shows that there is a very close relationship between these two approaches to constrained bimatrix games.

The Arrow/Debreu Theorem will be the starting point for our considerations in the paper. We will formulate this theorem in an equivalent form, better suited to the way we present our results. We begin with some notion.

A (*noncooperative*) *n-person game with constraints* is a quadruple

$$\mathbb{G} =< N, \{X_i\}_{i\in N}, \{S_i\}_{i\in N}, \{F_i\}_{i\in N} > \qquad (1)$$

where $N = \{1, 2, \ldots, n\}$ is a set of players, and for each $i \in N$
(a) X_i is a space of pure strategies of player i;
(b) S_i is a multifunction from the set $X_{-i} = \prod_{k\neq i} X_k$ to subsets of the set X_i. $S_i(x_{-i})$ is the set of pure strategies of player i admissible when the remaining players act according to x_{-i};
(c) $F_i : \prod_{k\in N} X_k \to R$ is the payoff function of player i.
Such a game with constraints will be also called a *constrained game.*

A *multistrategy* $x^* = (x_1^*, x_2^*, \ldots, x_n^*) \in \prod_{i=1}^n X_i$ is a (*pure*) *equilibrium* in n-person constrained game $\mathbb{G}$ of the form (1) if for all $i \in N$, $x_i^* \in S_i(x_{-i}^*)$ and

$$F_i(x^*) = max_{y_i\in S(x_{-i}^*)} F_i(x_{-i}^*, y_i). \qquad (2)$$

Now the Arrow/Debreu Theorem can be written in the folowing equivalent form.

THEOREM 1 *Let $\mathbb{G}$ be n-person game with constraints of the form (1), $n \geq 2$. Assume that (a) all pure strategy spaces X_i are convex and compact subsets of a Euclidean space R^{k_i}, (b) each payoff function F_i is continuous on $\prod_{i=1}^n X_i$ and concave with respect to i-th variable, and (c) all the multifunctions S_i are continuous and have nonempty, closed, convex values.*
Then the game $\mathbb{G}$ has a pure equilibrium.

The main purpose of the paper is to find a discrete counterpart of the Arrow/Debreu Theorem for two-person non-zero-sum games with finite sets X_1 and X_2 of players' pure strategies. Such games can be defined in a form of *constrained bimatrix games.* To this end, we extend the classical notion of convexity and concavity to finite sets and to functions with finite domains, respectively, to get discrete counterparts of assumptions of that theorem.

2. Preliminary definitions and results In the paper we will mainly concentrate on constrained games of the form (1) with finite pure strategy spaces X_i. Such games will be called *finite constrained games*, and the players' pure strategy spaces will be of the form

$$X_i = \{1, 2, \ldots, k_i\}, \qquad 1 \leq i \leq n. \qquad (3)$$

siderations to mixed equilibria in constrained games.

A *mixed strategy* of player i in game $\mathbb{G}$ is a probability distribution μ_i on the set X_i. Let us denote $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ and $\mu_{-i} = (\mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \dots, \mu_n)$. For arbitrary vector μ of players' mixed strategies, we put

$$F_i(\mu) = \int_{x \in X} F_i(x) d\mu_1 d\mu_2 \dots d\mu_n \quad \text{and} \quad S_i(\mu_{-i}) = \bigcap_{x_{-i} \in supp(\mu_{-i})} S_i(x_{-i}).$$

A (mixed) multistrategy μ is *admissible* when the last equality holds for all $i \in N$.

DEFINITION 1 *A multistrategy μ of players' mixed strategies $\mu = (\mu_1^*, \mu_2^*, \dots, \mu_n^*)$ is called a (mixed)* equilibrium *in constrained game $\mathbb{G}$, if for each player $i \in N$,* $\text{supp}(\mu_i^*) \subset S_i(\mu_{-i}^*)$ *and* $F_i(\mu^*) = \max_{y_i \in S(\mu_{-i}^*)} F(\mu_{-i}^*, y_i)$.

Now notice that if we take S_i in (1) as satisfying $S_i(x_{-i}) = X_i$ for all x_{-i}, then the constrained game $\mathbb{G}$ becomes a (classical) non-cooperative n-person game whose normal form can be simply written as

$$\Gamma = < N, \{X_i\}_{i \in N}, \{F_i\}_{i \in N} > . \tag{4}$$

It is trivially seen that the set of equilibria in such constrained game coincides with the set of its Nash equilibria.

Now we give definitions concerning several properties of finite games and players' strategies. All these notions are basic for our further considerations. Let Γ denote an n-person finite game described by (4) and (3).

DEFINITION 2 *A payoff function F_i of player i in game Γ, $i \in N$, is called* concave *with respect to i-th variable if for $j = 1, 2, \dots, n$ there exist strictly increasing sequences $y^j = (y_1^j, y_2^j, \dots, y_{k_j}^j)$ in $[0,1]$, and if there exists a function $f(y_1, \dots, y_n)$ from $[0,1]^n$ to $\mathbb{R}$, concave with respect to variable y_i and such that for all $(x_1, x_2, \dots, x_n) \in \prod_{i=1}^n X_i$, $f(y_{x_1}^1, y_{x_2}^2, \dots, y_{x_n}^n) = F_i(x_1, x_2, \dots, x_n)$.*

Game Γ is called concave if for each $i \in N$, payoff function F_i is concave with respect to variable x_i.

REMARK 1 There is a simple procedure verifying the concavity of a finite game Γ.

DEFINITION 3 *A mixed strategy μ of player i in game Γ is called two-adjoining-pure if it is of the form $\mu_i = \alpha\delta_a + (1-\alpha)\delta_{a+1}$ for some $0 \le \alpha \le 1$ and $a \in X_i$, $1 \le i \le n$. (Here δ_t denotes a degenerate probability distribution concentrated at point t.)*

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ $(m, n \ge 1)$ be payoff matrices of players 1 and 2 in a non-zero-sum two-person game Γ of the form (4) with

$$X_1 = \{1, 2, \dots, m\}, \qquad X_2 = \{1, 2, \dots, n\} \tag{5}$$

and

$$a_{ij} = F_1(i,j), \qquad b_{ij} = F_2(i,j) \qquad \text{for } 1 \le i \le m, \ 1 \le j \le n. \tag{6}$$

Let $\Gamma(A,B)$ denote a bimatrix game described by (5) and (6). We will also use the notation $\Gamma(A,B)_{m \times n}$ for $(m \times n)$-game $\Gamma(A,B)$ to emphasize the size of payoff matrices in it.

3. Main results In this section we will consider constrained bimatrix games. We formulate here (without proofs) our two main results (Theorems 2 and 3) about the existence equilibria in constrained bimatrix games. Their proofs will be given in the next section.

Let $m, n \ge 1$ be arbitrarily fixed natural numbers. Instead of the form (1), it is more convenient to describe such constrained games in the following equivalent form:

$$\mathbb{G} = \mathbb{G}(A, B, E_A, E_B) \tag{7}$$

where $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are payoff matrices for players 1 and 2 respectively, with $a_{ij} = F_1(i,j)$ and $b_{ij} = F_2(i,j)$ for $1 \le i \le m$, $1 \le j \le n$. Here E_A and E_B are nonempty sets of pairs of players' strategies (i,j), admissible for players 1 and 2 in game $\mathbb{G}$, respectively, that is, $(i,j) \in E_A \iff i \in S_1(j)$ and $(i,j) \in E_B \iff j \in S_2(i)$.

To formulate our main result about the existence of a mixed equilibrium in constrained bimatrix game $\mathbb{G}$ of the form (7), we need to assume the following three conditions.

Z1 (symmetry) $E_A = E_B \neq \emptyset$.

Z2 (sections convexity) (a) If $0 \le i \le k \le l \le m$ and $(i,j) \in E_A$ and $(l,j) \in E_A$ for some $j \in X_2$, then $(k,j) \in E_A$.

(b) If $0 \le j \le k \le l \le n$ and $(i,j) \in E_B$ and $(i,l) \in E_B$ for some $i \in X_1$, then $(i,k) \in E_B$.

Z3 (game convexity) There exists a concave bimatrix game $\Gamma(A^*, B^*)_{m \times n}$ such that $a_{ij} = a^*_{ij}$ for all pairs $(i,j) \in E_A$, and $b_{ij} = b^*_{ij}$ for all pairs $(i,j) \in E_B$.

Assumption **Z1** does not have a counterpart in the Arrow/Debreu Theorem. Assumption **Z2** is a discrete counterpart of the assumption about the convexity of images of multifunctions S_i in the Arrow/Debreu Theorem, and assumption **Z3** is a discrete counterpart of the assumption of concavity of payoff functions considered there.

Now we are ready to formulate our first main result.

THEOREM 2 *If a constrained bimatrix game $\mathbb{G}(A, B, E_A, E_B)$ satisfies assumptions* **Z1–Z3**, *then it has an equilibrium consisting of two two-adjoining-pure strategies.*

REMARK 2 One can show that after removing assumption **Z3**, constrained bimatrix game $\mathbb{G}(A, B, E_A, E_B)$ can have equilibrium different from that of announced in Theorem 2. But the authors do not know if assumptions **Z1** and **Z2** guarantee the existence of an equilibrium in $\mathbb{G}$ at all. The more general and very interesting question is what necessary and sufficient conditions guarantee the existence of equilibria in constrained bimatrix games.

Apart from constrained bimatrix games, one can consider generalized bimatrix games (without constraints) with possible payoffs equal to $-\infty$. It is obvious that multistrategies generating such payoffs are exeptionally disadvantageous for the players. One can see here some similarity between forbidden (not admissible) multistrategies and multistrategies generating payoffs $-\infty$ in those games. It appears that, in fact, there is a very close relationship between those two types of games. This will be described below in Theorem 3 which is our second main result. To formulate it we need to consider the additional (not very restrictive) assumption **Z4**.

Let $\mathbb{G} = \mathbb{G}(A, B, E_A, E_B)$ be a constrained bimatrix game. Let us associate with $\mathbb{G}$ a new generalized bimatrix game defined as $\Gamma_{\mathbb{G}} = \Gamma(A_{\mathbb{G}}, B_{\mathbb{G}})_{m \times n}$, where payoff matrices $A_{\mathbb{G}} = [a_{ij}]$ and $B_{\mathbb{G}} = [b_{ij}]$ are described by

$$a_{ij} = \begin{cases} a_{ij} & \text{for} \quad (i,j) \in E_A \\ -\infty & \text{for} \quad (i,j) \notin E_A \end{cases} \qquad b_{ij} = \begin{cases} b_{ij} & \text{for} \quad (i,j) \in E_B \\ -\infty & \text{for} \quad (i,j) \notin E_B \end{cases} .$$

(for game $\Gamma_{\mathbb{G}}$ we use the convention, $0 \cdot (-\infty) = 0$).

Z4 (full section) There exists $j \in \{1, 2, \ldots, n\}$ such that for each $i \in \{1, 2, \ldots, m\}$ the pair $(i,j) \in E_B$, or there exists $k \in \{1, 2, \ldots, m\}$ such that for each $l \in \{1, 2, \ldots, n\}$, the pair $(k,l) \in E_A$.

THEOREM 3 *Assume that constrained game $\mathbb{G} = \mathbb{G}(A, B, E_A, E_B)$ satisfies assumptions* **Z1–Z3**. *Then the generalized bimatrix game $\Gamma_{\mathbb{G}}$ has a Nash equilibrium in two-adjoining-pure strategies with finite payoffs for both players. Moreover, each equilibrium in $\mathbb{G}$ is a Nash equilibrium in game $\Gamma_{\mathbb{G}}$.*

If additionally assumption **Z4** *holds, then the set of all equilibria in $\mathbb{G}$ coincides with the set of all Nash equilibria in $\Gamma_{\mathbb{G}}$.*