

From Lemke-Howson to Kakutani: Extended Abstract

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Abstract

We give a full proof of the Kakutani (1941) fixed point theorem that is brief, elementary, and based on game theoretic concepts. An *imitation game* is a finite two person normal form game in which the strategy spaces for the two agents are the same and the goal of the second player is to choose the same strategy as the first player. A simplified version of the algorithm of Lemke and Howson (1964) establishes that every imitation game has a Nash equilibrium. Kakutani's theorem is obtained by defining certain imitation games whose Nash equilibria yield approximate fixed points.

We give a new proof of Kakutani's fixed point theorem. In comparison with earlier proofs of Brouwer's and Kakutani's fixed point theorems (to a greater or lesser extent depending on the proof) our argument has several advantages. It is simple and elementary. It is direct, arriving at Kakutani's theorem without an intermediate stop at Brouwer's theorem. It is based on game theoretic concepts and reasoning, so it is complementary to the goals of instruction in theoretical economics. It leads to interesting algorithms for computing approximate fixed points.

A finite two player normal form game is an *imitation game* if the sets of pure strategies for the two players are the same and the payoff of agent 2 is one or zero according to whether she plays the same pure strategy as agent 1. The first major step in our argument is the demonstration that every imitation game has a Nash equilibrium. The Lemke-Howson (Lemke and Howson (1964)) algorithm gives an elementary proof of existence of Nash equilibrium for two person normal form games; we contribute the observation that this procedure has a particularly simple form when applied to imitation games.

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As a simplified version of the Lemke-Howson algorithm, it shares many of the properties of its parent. It is possible to use it to find more than one equilibrium, first of all because multiple starting points are possible, but also because, having found an equilibrium from one starting point, it is possible to run the algorithm in reverse along the graph corresponding to a different starting point. Using lexicographic methods, it is possible to develop more sophisticated versions that do not depend on a nondegeneracy assumption.

The authors have also discovered a different proof of the existence of Nash equilibrium for an imitation game that is based on a quite different algorithm.

We now explain how this result implies Kakutani's fixed point theorem. Suppose that C is a nonempty compact convex subset of a Hilbert space and $F : C \rightrightarrows C$ is an upper semicontinuous correspondence. Let $D \subset C$ be nonempty and finite, and let $f : D \rightarrow C$ be a function. Define an imitation game $(S, T; u, v)$ by letting $S = T = D$ and letting $u, v : S \times T \rightarrow \mathbb{R}$ be the functions $u(s, t) = -\|s - f(t)\|^2$ and $v(s, t) = \delta_{st}$. (Here δ_{st} is the Kronecker delta.) Let (σ, τ) be a mixed Nash equilibrium of this game, and let $\omega := \sum_{t \in T} \tau(t)t$. A simple calculation shows that the support of σ is contained in the set of points in S that are closest to ω , and of course the support of τ is contained in the support of σ . If $f(d) \in F(d)$ for all $d \in D$, then ω is an approximate fixed point of F . An exact fixed point is obtained by taking a convergent subsequence of the sequence of approximate fixed points obtained from the Nash equilibria of a sequence of imitation games derived from a sequence $\{(D^r, f^r)\}$ with

$$\lim_{r \rightarrow \infty} \max_{x \in C} \min_{d \in D^r} \|x - d\| = 0.$$

Consider the following algorithm for computing an approximate fixed point of a continuous function $f : C \rightarrow C$. Choose $x_0 \in C$ arbitrarily. Assuming that $D^r := \{x_0, x_1, \dots, x_r\} \subset C$ has been determined, compute a mixed equilibrium (σ^r, τ^r) of the imitation game $(S^r, T^r; u^r, v^r)$ derived as above from D^r , and let $x_{r+1} := \omega^r := \sum_{t \in T^r} \tau^r(t)t$. Any limit point of the sequence $\{x^r\}$ must be a fixed point of f . In comparison with earlier algorithms for computing approximate fixed points, this procedure has several advantages. It is simple and flexible. There is no need for the computer to “know” about a simplicial subdivision of C , and consequently no restriction on the geometry of C . Algorithms based on simplicial subdivision require complicated refinements to handle the problem of “restart:” having computed a coarse approximation, the basic versions of these algorithms provide no method for improving its precision aside from redoing the entire calculation with a finer simplicial subdivision. But in computing (σ^r, τ^r) one may start the simplified Lemke-Howson algorithm at x^r , which should typically result in a relatively quick computation of x^{r+1} .

References

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