On Indifference Curves in Mean-Risk Decision Models

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In many decision models under uncertainty risky prospects are represented by a family of distributions which differ from one another only by location and scale parameters, i.e., there exists a distibution F of the family, called its generator such that any distribution G of the family has the form

$$G(x) = F(\alpha x + \beta)$$

for $x \in \mathbb{R}$ where $\alpha > 0$ and $\beta \in \mathbb{R}$. These parameters depend on the vector of decision variables. In the literature the family of such distributions is called a linear distribution class, location-scale family of distributions or one says that the family satisfies the location and scale parameter condition (the LS condition for short). For example, such a family corresponds a two-parameter family of risky prospects depending in the positive affine way on a certain risky prospect, called a generator of the family, i.e., any risky prospect Y of the family has the form

$$Y = \alpha X + \beta, \tag{1}$$

where $\alpha > 0$ is a scale parameter, $\beta \in \mathbb{R}$ a location parameter and X a generator. These parameters depend on the vector of decision variables. In this way one can obtain any family of distributions satisfying the LS condition. An agent has an increasing concave utility function and he maximizes the expected utility of a risky prospect as the function of the vector of decision variables. In the literature one can encounter many such decision models, e.g., in: portfolio theory initiated by Markowitz, the Capital Asset Pricing Model of Sharpe and Lintner and Sandmo's model of the competitive firm under price uncertainty. We will only consider families of distributions satisfying the LS condition which are non-degenerate and have means. Any such a family contains a generator with mean equal to zero. Moreover, if there exists $p > 1$ such that any distribution of the family has finite absolute moment of order p, then this generator can be uniquely chosen in such a way that, additionally, its absolute moment of order p equals one. If this family corresponds (1), then this generator corresponds

$$(Y - E(Y))/||Y - E(Y)||_p,$$

where Y is any risky prospect of the family (1) and $||\cdot||_p$ is the L_p-norm. If in (1) one chooses X as this generator, then

$$E(Y) = \beta \quad \text{and} \quad ||Y - E(Y)||_p = \alpha.$$

Therefore in this case α may be considered as the measure of risk associated with the risky prospect (1). If $p = 2$, then α is the classical measure of risk - the standard deviation. In any case the location parameter β (provided that $E(X) = 0$) and the scale parameter α may be also considered as the mean and the measure of risk of the risky prospect (1), respectively. They will be denoted by μ and r, respectively. Then the expected utility of a risky prospect is only a function of its risk and mean. The theorems on properties of this function and its level curves, called indifference curves, go back to Tobin. We present them in a novel way. Moreover, we show that indifference curves have no vertical asymptotes in the (r, μ) - plane; this has not been established to date. The purpose of the talk is to fill this gap. The last property means that an agent cannot obtain unbounded mean with bounded risk preserving the same level of the expected utility of risky prospects. An application of this property to a proof of the theorem on the existence of equilibrium in the CAPM, in a particular case, is given.