

# Production Games, Core Deficit, and Shadow Prices

S. D. FLÅM\*

July 7, 2004

ABSTRACT. This note considers production (or market) games with transferable utility. Prime objects are explicitly computable core solutions, or somewhat "deficit" versions of such, fully defined by shadow prices. Main arguments revolve around standard Lagrangian duality. A chief concern is to relax, or avoid, the commonplace assumption that all preferences and production possibilities be convex. Thus we obtain novel results about non-emptiness of the core - and about specific imputations therein.

## 1. INTRODUCTION

Considered below are cooperative  $TU$ -games among parties concerned with equitable sharing of efficient production costs. There is a nonempty finite player set  $I$ , and coalition  $S \subseteq I$  would incur cost  $C_S \in \mathbb{R} \cup \{+\infty\}$  if going alone. Reasonably assume  $C_I \leq \sum_{i \in I} C_i < +\infty$ . A cost profile  $c = (c_i) \in \mathbb{R}^I$  is then declared in the *core*, as signaled by writing  $c \in \text{core}$ , iff it embodies

$$\begin{aligned} \text{full cost cover:} & \quad \sum_{i \in I} c_i \geq C_I \quad \text{and} \\ \text{coalitional stability:} & \quad \sum_{i \in S} c_i \leq C_S \quad \text{for each nonempty subset } S \subseteq I. \end{aligned}$$

Plainly, this solution concept makes good sense when *core* is neither empty, nor too large, nor very sensitive to data. Sensitivity can be mitigated by replacing each "participation constraint"  $\sum_{i \in S} c_i \leq C_S$  with the corresponding relaxed form:  $\sum_{i \in S} c_i \leq C_S + \varepsilon |S|$ ,  $\varepsilon > 0$ . Such relaxation, while yielding so-called  $\varepsilon$ -approximate cores [2], is quite convenient, but not always most natural.<sup>1</sup> In fact, regarding possible emptiness of the core, note that coalitional stability comes easily by charging each player too little payment. So, the real bite resides in the requirement that the overall charge  $\sum_{i \in I} c_i$  cover the total cost  $C_I$ .

This simple observation indicates that valuable insights might sometimes be gained by relaxing merely the full cover condition to read instead:

$$\sum_{i \in I} c_i \geq C_I - D. \tag{1}$$

Here the monetary entity  $D \geq 0$  stands for a *deficit*. For

---

\*Economics Department, Bergen University 5007 Norway; sjur.flaam@econ.uib.no. I thank IN-DAM for generous support and Universita degli Studi di Trento for great hospitality.

<sup>1</sup>In particular, why should the perturbation be "uniform" across coalitions? And how does  $\varepsilon$  relate to underlying data?

**Example 1:** Consider three producers, each obliged to put out the amount  $e_i = 1/2$  of the same homogenous commodity. The cost functions  $f_i : \mathbb{X}_i \rightarrow \mathbb{R}$  are equal and defined by  $\mathbb{X}_i = [0, 1]$  and  $f_i(x_i) :=$  the smallest integer  $\geq x_i$ . Coalition  $S$  incurs cost

$$C_S := \inf \left\{ \sum_{i \in S} f_i(x_i) : \sum_{i \in S} x_i = \sum_{i \in S} e_i \right\}. \quad (2)$$

Clearly,  $C_S = 1$  when  $|S| \leq 2$ . Since  $C_I = 2$  is "too large", the core is empty. Substitution  $C_I \leftarrow C_I - d$  with *minimal deficit*  $d = 1/2$  suffices to have  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$  as the single core element. Alternatively, replacing  $f_i$  with the largest convex function  $\text{conv}f_i \leq f_i$  in (2), we get  $\text{conv}f_i(x_i) = x_i$ , and costs  $S \mapsto |S|/2$ , these admitting  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$  as unique core element. However, letting  $\mathbb{X}_i = \{0, 1\}$  yields coalitional cost  $= 1$  if  $|S| = 2$ ,  $+\infty$  otherwise, and again  $\text{core} = \emptyset$ .  $\square$

For simplicity,  $c = (c_i) \in \mathbb{R}^I$  is said to be in the  $D$ -core of the cost sharing game iff it satisfies (1) and  $\sum_{i \in S} c_i \leq C_S \forall S \subseteq I$ . The set so defined is a *polyhedron* (i.e. an intersection of closed half-spaces). Further, the defining constraints imply  $C_I - D - \sum_{j \neq i} C_j \leq c_i \leq C_i$ , so  $D$ -core must be bounded - whence a *polytope* (i.e. the convex hull of finitely many points, if any). Since  $D$ -core decreases with  $D$ , the smallest possible deficit is worth closer scrutiny. In this regard, it holds

**Proposition 1.** (On the minimal deficit) *Let  $d := \inf \{D : D\text{-core} \neq \emptyset\}$  denote the **minimal deficit**. Then  $d$ -core is nonempty, and*

$$d\text{-core} = \cap \{D\text{-core} : D > d\}.$$

Moreover, in terms of linear programming,

$$\begin{aligned} C_I - d &= \max \left\{ \sum_{i \in I} c_i : \sum_{i \in S} c_i \leq C_S \forall S \subseteq I \right\} \\ &= \min \left\{ \sum_S w_S C_S : w_S \geq 0 \forall S \text{ and } \sum_{S: i \in S} w_S = 1 \forall i \right\}. \quad \square \end{aligned}$$

Proposition 1 might lure one into thinking that, in principle, no more need be said.<sup>2</sup> Often though, at least *four* queries come up here. *First*, both linear programs here above presume explicit knowledge of all costs  $C_S$ , in total  $2^{|I|} - 1$  of them. Typically, to generate these numbers is quite a task. So, an easier approach would be welcome. *Second*, it appears worthwhile to assess, or estimate,  $d$  in terms of primitive data, if any, underlying the characteristic function  $S \mapsto C_S$ . *Third* - for the sake of computation, interpretation or implementation - it is desirable to display *explicit* core

<sup>2</sup>In particular, as corollary we recover the Bondareva-Shapley [?] result:  $d = 0 \Leftrightarrow \text{core is non-empty} \Leftrightarrow \text{the game is balanced} \Leftrightarrow C_I \leq \sum_S w_S C_S$  for all balanced families  $(w_S) \geq 0$  of weights.

solutions or approximate versions of such, defined via given data. *Fourth* and finally, if  $d > 0$  - that is, if the game features some "imbalance" - one would like to see exactly where and why.

This note opens up a vista on all these issues as they relate to so-called *production* (or market) *games*. Such games are somewhat special, but of frequent occurrence and great importance. References include ...

The objectives and novelties are several here below: First, after recalling that the minimal deficit is inferior to the much studied *duality gap* in optimization, we use related theory to provide a new and weak condition, concerning *weak stability*, which entails  $d = 0$  and *core*  $\neq \emptyset$ . Second, provided the production game be *stable*, we shall exhibit explicit core solutions, generated by so-called shadow prices.

Motivation for this inquiry stems from the need to reach beyond instances with convex preferences and production sets. Notably, there should be ample room for discrete activity (decision) sets - and for non-divisibility of various selected inputs. What imports is that some key resources - possibly modulo time-shared use - be perfectly divisible and transferable. Bundles of these resources are privately owned. Their scarcity generates common willingness to pay for appropriation. Thus emerge endogenous shadow prices. In their turn these determine specific core imputations - albeit only up to deficit  $D \geq d$ .

Section 2 defines the games and includes a running example. Section 3 reviews some non-standard, novel results on Lagrangian duality that prove helpful. Section 4 contains the main results, believed to interest several sorts of readers. In particular, we address mathematicians, more concerned with optimization

#### REFERENCES

- [1] A. Bourass and E. Giner, Kuhn-Tucker conditions and integral functionals, *J. Convex Analysis* 8, 2, 533-553 (2001).
- [2] R. B. Myerson, *Game Theory*, Harvard University Press, Cambridge (1991).
- [3] D. Samet and Zemel, On the core and dual set of linear programming games, *Mathematics of Operations Research* 9(2), 309-316 (1994).
- [4] M. Sandmark, Production games under uncertainty, *Computational Economics* (1999).
- [5] L. S. Shapley and M. Shubik, On market games, *J. Economic Theory* 1, 9-25 (1969).
- [6] G. Owen, On the core of linear production games, *Mathematical Programming* 9, 358-370 (1975).