# CONVEXITY of PRODUCTION, COMMON POOL and OLIGOPOLY GAMES Extended abstract *

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A *cooperative game* with transferable utility (TU) is a pair $\langle N, v\rangle$, where $N$ is a nonempty, finite set and $v : 2^N \to \mathbb{R}$ is a *characteristic function*, defined on the power set of $N$, satisfying $v(\emptyset) := 0$. An element of $N$ (notation: $i \in N$) and a nonempty subset $S$ of $N$ (notation: $S \subseteq N$ or $S \in 2^N$ with $S \neq \emptyset$) is called a *player* and *coalition* respectively, and the associated real number $v(S)$ is called the *worth* of coalition $S$. Concerning the modelling part of game theory, it is customary to investigate whether or not any appropriate class of cooperative TU games satisfies one or another appealing property. Without going into details, we state that the so-called *convexity property* of the characteristic function $v$ is a widely-spread concept through the game theory literature. Any cooperative TU game $\langle N, v\rangle$ is said to be a *convex game* (cf. Shapley, [6], 1971) if it holds $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subseteq N$ or equivalently, for all $i \in N$, $j \in N$, $i \neq j$, and all $S \subseteq N\backslash\{i, j\}$, it holds

$$v(S \cup \{i\}) + v(S \cup \{j\}) \leq v(S) + v(S \cup \{i, j\}) \qquad (0.1)$$

With reference to a linear inverse demand function, Zhao ([7], 1999) presented necessary and sufficient conditions for the convexity of so-called oligopoly games with transferable technologies (that is, every firm in a coalition can produce according to the cheapest technology available in the coalition). Norde et al. ([4],2002) and Pham Do ([5], 2003, Ph.D. Thesis) proved that linear oligopoly games without transferable technologies (but with linear cost functions) are convex games, but their convexity proof is extremely exhaustive, requires a lot of mathematical notation, and its proof technique is based on some complicated induction step to be applied to a slightly adapted oligopoly game. Our main purpose is to highlight the uniform approach to establish a new proof of the convexity property for linear oligopoly games without transferable technologies similar to the already existing proofs of the convexity property for the so-called common pool game (cf. Meinhardt, [2], 2000; Driessen and Meinhardt, [1], 2001) as well as the so-called simple production game (cf. Moulin, [3], 1990). The

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essential part of the forthcoming proof technique of convexity for such classes of TU games concerns the *interchangeability* of both players $i$ and $j$ with respect to the convexity condition (0.1). That is, this convexity condition (in terms of marginal contributions) does not change whenever player $i$ is replaced by player $j$ and vice versa. Generally speaking, let $N$ be a finite set of firms.

**Theorem 0.1.** (cf. [3]) Let $u_i : \mathbb{R}_+ \to \mathbb{R}_+$ be firm $i$'s *utility function* (with reference to any *production level* denoted by a non-negative variable $x \in \mathbb{R}_+$). Consider a simple production economy $\langle N, (u_i)_{i\in N}, c\rangle$ consisting of firm $i$'s utility function $u_i$, $i \in N$, and a single cost function $c : \mathbb{R}_+ \to \mathbb{R}_+$. The worth $v(S)$ of any coalition $S$ of firms is determined by some production level that maximizes the sum of members' utilities minus its cost of production. The characteristic function $v$ of the associated *simple production economy game* $\langle N, v\rangle$ is defined by $v(\emptyset) = 0$ and

$$v(S) = \max_{x\geq 0}\left[\sum_{k\in S} u_k(x) - c(x)\right] \quad \text{for all } S \subseteq N,\ S \neq \emptyset. \tag{0.2}$$

Without any further assumptions on the cost function and the weakly increasing utility functions, the simple production economy game $\langle N, v\rangle$ of (0.2) is a convex game (i.e., (0.1) holds).

**Theorem 0.2.** (cf. [4], [5]) Let $w_i > 0$ and $c_i \geq 0$ respectively denote firm $i$'s *capacity* of production, and *marginal cost*. Consider a *linear oligopoly situation* $\langle N, (w_i)_{i\in N}, (c_i)_{i\in N}, a\rangle$ composed of a linear inverse demand function $p : \mathbb{R}_+ \to \mathbb{R}_+$ with the the intercept $a \geq 0$ and linear cost functions. That is, $p$ is given by $p(z) := \max[0,\ a - z]$ for all $z \geq 0$. For any $S \subseteq N$, $S \neq \emptyset$, denote by $\Pi_S := \Pi_{k\in S}[0,\ w_k]$ the production space of $S$ and write any *feasible production schedule* $\vec{x}^S := (x_k)_{k\in S} \in \Pi_S$. The characteristic function $v$ of the associated *linear oligopoly game* $\langle N, v\rangle$ is defined by $v(\emptyset) = 0$ and

$$v(S) = \max_{\vec{x}^S\in\Pi_S}\left[\left[a - \sum_{k\in N\setminus S} w_k - \sum_{k\in S} x_k\right]\cdot\sum_{k\in S} x_k - \sum_{k\in S} c_k\cdot x_k\right] \quad \text{for all } S \subseteq N,\ S \neq \emptyset. \tag{0.3}$$

Without any further assumptions, the oligopoly game $\langle N, v\rangle$ of (0.3) with a linear inverse demand function and linear cost functions is a convex game (i.e., (0.1) holds).

Let us now outline the *uniform approach* to investigate the convexity condition (0.1) for such classes of games arising from maximization problems. Consider the underlying game model $\langle N, v\rangle$. Let $i \in N$, $j \in N$, $i \neq j$, and $S \subseteq N\setminus\{i, j\}$. Concerning the left hand side, describe the worth of both augmented coalitions $S \cup \{i\}$ and $S \cup \{j\}$ in terms of their corresponding (yet unknown) *maximizers* (deduced from the objective functions of their corresponding maximization problems). Concerning the right hand side, *underestimate* the worth of both coalitions $S$ and $S\cup\{i, j\}$ by evaluating the objective function of the two corresponding maximization problems at *appropriately chosen adaptations of the former two maximizers*. The more complex the game model, the more complex to guess such adaptations of the former two maximizers. That is, at the first stage, we replace the convexity condition (0.1) for the characteristic function $v$ by one sufficient condition with reference to four objective functions, each of one evaluated at (appropriately chosen adaptations of) their maximizers. At the second stage, exploit, as far as possible, the interrelationships among the various objective functions to simplify the former sufficient condition with reference to the objective functions. At the third and final stage, claim that the simplified sufficient condition holds true due to the interchangeability of both players $i$ and $j$ with respect to the convexity condition (0.1). Because of $v(\emptyset) = 0$, it is necessary to present a separate proof whenever the coalition $S$ equals the empty set, but fortunately, this separate proof is much less complex.

As an introductory example, consider the setting of the simple production economy game $\langle N, v\rangle$ of (0.2). For all $T \subseteq N$, $T \neq \emptyset$, let $y_T \geq 0$ denote a maximizer for the maximization problem (0.2) with reference to coalition $T$, that is $v(T) = \sum_{k \in T} u_k(y_T) - c(y_T)$.
Let $i \in N$, $j \in N$, $i \neq j$, and $S \subseteq N \backslash \{i, j\}$. The two maximizers $y_{S\cup\{i\}}$ and $y_{S\cup\{j\}}$ respectively for the maximization problems (0.2) with reference to the two augmented coalitions $S \cup \{i\}$ and $S \cup \{j\}$ respectively may be considered as feasible production levels for the maximization problems (0.2) with reference to the two coalitions $S$ and $S \cup \{i, j\}$ respectively in order to underestimate the worth $v(S)$ and $v(S \cup \{i, j\})$. As a result, we replace the convexity condition (0.1) for the simple production economy game $v$ by the following weaker, but sufficient condition in which the single cost function cancels out, while only one utility function is left: $u_i(y_{S\cup\{i\}}) \leq u_i(y_{S\cup\{j\}})$. At this second stage, we exploit that any utility function is supposed to be weakly increasing, so that the former inequality is equivalent to $y_{S\cup\{i\}} \leq y_{S\cup\{j\}}$. Finally, at the third and final stage, we claim, without loss of generality, that the essential inequality $y_{S\cup\{i\}} \leq y_{S\cup\{j\}}$ holds due to the interchangeability of both players $i$ and $j$ with respect to the convexity condition (0.1). The remaining case $S = \emptyset$ is left to the reader.

We claim that the convexity property for linear oligopoly games and common pool games as well can be established along similar reasonings (using more advanced choices/calculations)

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