Arrow-Hahn economic models with weakened conditions of continuity

Inese Bula and Dace Rika,
Department of Mathematics, University of Latvia,
Rainis boulv. 19, Rīga, LV-1586, LATVIA
E-mail: buls@lanet.lv, dace.rika@vidm.gov.lv

One of the basic assumption in mathematical modelling of the equilibrium economc model is the continuity of functions describing the model. This article analyses Arrow-Hahn economic model [1] by weakening the condition of continuity of excess demand functions and proving existence of quasi-equilibrium in this new model.

At first we give the short description of a economic model considered by Arrow and Hahn in [1].

Let there be n ($n \in \mathbb{N}$) different goods (commodities) on the market and a finite number of economic agents: households and firms. The excess demand of good i is defined as $x_i - y_i - \overline{x}_i$, where x_i is the total demand of good i, y_i is the total supply of good i and $\overline{x}_i$ is the resources of good i, $i = 1, ..., n$.

Assumption F. Let $p = (p_1, ..., p_n)$ be n-dimensional price vector with the prices p_i for one unit the good i, $i = 1, 2, ..., n$. For any p let the excess demand for i be characterized by a unique number $z_i(p)$ and so the unique vector $z(p) = (z_1(p), \ldots, z_n(p))$ - the excess demand function with excess demand functions for i as components ($i = 1, 2, ..., n$) - is well defined.

Assumption H. $\quad z(p) = z(\lambda p), \quad \forall p > \mathbf{0}$ and $\lambda > 0$.

From the Assumption H follows that prices can be normalized (see [1], p.20 or [5], p.10). Therefore, further on we consider only prices from the $(n-1)$-dimensional simplex of R^n $S_n = \{p = (p_1, p_2, ..., p_n) \mid p_i \geq 0 \text{ and } \sum_{i=1}^{n} p_i = 1\}$.

Assumption W or Walras' Law. $\quad p\, z(p) = 0, \quad \forall p \in S_n$.

Assumption C. The excess demand function z is continuous on its domain of definition S_n.

A price $p^* \in S_n$ is called an **equilibrium** (price) if $z(p^*) \leq 0$.

For the standard model of an economy with a finite number of goods and agents such prices always exist ([1]).

From the Assumption C implies that the demand for free goods is bounded. "We shall want to weaken this restriction for several reasons: it is not unreasonable that demand for at least some goods might approach infinity as the price approaches zero; the non-satiation hypothesis that underlies Walras' law it at least partly inconsistent with satiation in any single good; the assumption that all goods are gross substitutes,..., implies that demands may approach infinity as prices go to zero" (we cite [1], P.29). Therefore we will assumed that the sum of excess demands approaches infinity whenever excess demand is undefined. We introduce another two assumptions.

Assumption B. There exists a positive number B such that for all $p \in S_n$: $z_i(p) > -B$ for all $i = 1, 2, \ldots, n$, i.e. $z(p)$ is bounded from below.

Assumption C'. The excess demand function z is defined for $p \in S_n$ such that $p_i > 0$, $i = 1, 2, \ldots, n$, and possibly for other p and is continuous whenever defined. If z is not defined for p^0, then $\lim\limits_{p \to p^0} \sum\limits_{i=1}^{n} z_i(p) = +\infty$.

It is possible to show that in this changed situation equilibrium exist too ([1]).

Now we consider discontinuous excess demand functions. A class of discontinuous mappings is defined as follows. Let (X, d) and (Y, ϱ) be two metric spaces and w a positive number.

DEFINITION 1 ([2]). A mapping $f \colon X \to Y$ is said to be ***w*-discontinuous at the point** $x_0 \in X$ if for every $\varepsilon > 0$ there exists δ such that whenever $x \in X$ and $d(x_0, x) < \delta$ follows that $\varrho(f(x_0), f(x)) < \varepsilon + w$. A mapping f is called ***w*-discontinuous in** X if it is w-discontinuous at all points of X.

The notion of w-discontinuous maps is not new. It is already found in [6] as the concept of *oscillation* or in [4] as *continuity defect.* The notion of w-discontinuity (former w-continuity) was introduced by the first author in [2].

The following essential result is proved by O.Zaytsev in [7] and can be considered as a generalization of the Bohl-Brouwer-Schauder fixed point theorem for w-discontinuous mappings.

THEOREM 1 ([7]). Let K be a nonempty, compact and convex subset in a normed vector space X. For every w-discontinuous mapping $f : K \to K$ ($w > 0$) there exists a point $x^* \in K$ such that $\| x^* - f(x^*) \| \leq w$.

The model with w-discontinuous excess demand function in whole set S_n is analysed in [3]. But now we make assumption of continuity in following manner.

Assumption DC'. The excess-demand function z is defined for $p \in S_n$ such that $p_i > 0, i = 1, 2, \ldots, n$, and possibly for other p and is w-discontinuous whenever defined. If z is not defined for p^0 then

$$\lim_{p \to p^0} z_i(p) = \begin{cases} +\infty, & \text{if } p_i = 0 \\ \in \mathbf{R}, & \text{if } p_i \neq 0 \end{cases}, i = 1, 2, \ldots, n.$$

Let $S'_n = \{p \in S_n | z(p) \text{ is defined } \}$.

Since z is not defined in whole set S_n then small corection is necessary in Assumptions B.

Assumption B'. There exists a positive finite number B such that for all $p \in S'_n : z_i(p) > -B$ for all $i = 1, .2, \ldots, n$, i.e. $z(p)$ is bounded from below in the set S'_n.

It is quite natural that for every price vector $p \in S'_n$ there exist at least one good i with the price $p_i > 0$ and such that the demand for them is satisfied, i. e. $z_i(p) \leq 0$. Therefore for each $p \in S'_n$ the inequality $\gamma_p = \sum\limits_{z_i(p) \leq 0} p_i > 0$ is satisfied. Our next assumption requires the existence of a uniform lower bound for the sums $\sum_{z_i(p)\leq 0} p_i$ for all $p \in S_n'$.

Assumption Γ. $\gamma = \inf_{p\in S_n'} \gamma_p > 0.$

This Assumption Γ is independent on the Walras' Law (see [3]).

It seems to be clear that it would be hard to find out why an equilibrium exists in our model. But it will be possible if we can estimate the unsatisfied aggregate demand. This leads to the concept of quasi- or k-equilibrium.

DEFINITION 2. Let $k > 0$. A price vector $p^* \in S_n'$ is called a **k-equilibrium** if it satisfies the condition $\sum_{z_i(p^*)>0} z_i(p^*) \leq k$.

By there assumptions we can prove existence of quasi-equilibrium in mean of Definition 2.

THEOREM 2. Let for some economy with a finite number n of goods the Assumptions F, H, B', DC' and Γ be fulfiled. Let

$$w_+ = \frac{-(n+1)+\sqrt{(n+1)^2+4\min\{\frac{n}{n+1},\gamma\}}}{2n}.$$

If $w < w_+$ then there exists a k-equilibrium for each

$$k \geq \min\left\{n-1, \frac{nw^2+(n+1)w}{2\gamma-(nw^2+(n+1)w)}\right\}.$$

If $w = 0$ (excess demand function is continuous in the set S_n') then we obtain the classical equilibrium. But what does mean $w = 0.001$ or $w = 10^{-10}$?

References

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