On the distribution of dividends in games with ordered players*

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Abstract

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utility*, or simply a TU-game, being a pair (N, v), where N is a finite set of players and $v\colon 2^N \to \mathbb{R}$ is a characteristic function on N such that $v(\emptyset) = 0$. For any coalition $S \subseteq N$, $v(S)$ is the worth of coalition S, i.e. the members of coalition S can obtain a total payoff of $v(S)$ by agreeing to cooperate. In a TU-game players only differ with respect to their position in the game. Examples of models in which players not only differ with respect to their position in the game, but also are part of some relational structure (which possibly affects the cooperation possibilities or payoff distributions) are the *games in coalition structure* in which it is assumed that the set of players is partitioned into disjoint sets which represent social groups such that for a particular player it is more easy to cooperate with players in its own group than to cooperate with players in other groups, and the games with *limited communication structure* in which the edges of an undirected graph on the set of players represent binary communication links between the players such that players can cooperate only if they are connected.

The underlying paper is in line with *games with a permission structure* in which it is assumed that players in a TU-game are part of a hierarchical organization in which there are players that need permission from other players before they are allowed to cooperate within a coalition, see Gilles, Owen and van den Brink (1992), Gilles and Owen (1994), van den Brink and Gilles (1996) and van den Brink (1997). The hierarchical structure is represented by a directed graph or *digraph* (N, D) with $D \subseteq N \times N$ a binary relation on N.

In this paper we also assume the players to be hierarchically ordered and refer to a triple (N, v, D) as a *game with ordered players*. However, instead of restricting cooperation possibilities as done in games with a permission structure, we let the

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hierarchical order directly affect the distribution of *dividends* (see Harsanyi (1959)) in the game. We introduce two solutions which distribute dividends in the spirit of the *Selectope* (see Hammer, Peled and Sorensen (1977) and Derks, Haller and Peters (2000)) and *Harsanyi set* (see Vasil'ev (1978, 1981) and Vasil'ev and van der Laan (2002)) for arbitrary TU-games. For TU-games these solutions are identical and to any game they assign the set of payoff vectors that are obtained by distributing the dividend of every coalition among the players in the corresponding coalition in any possible way.

In the *Selectope for games with ordered players* we assign the dividend of a coalition to a player that is not dominated within the coalition. In the *Harsanyi set for games with ordered players* we distribute the dividend of a coalition among the players in that coalition in a way such that the share of a player that is dominated within the coalition is at most equal to the share of a player by which it is dominated in the coalition.

Although for TU-games the Selectope and Harsanyi set are equivalent, for games with ordered players their ideas give different solutions. For any game with ordered players the Selectope is a subset of the Harsanyi set. Further, the Harsanyi set for games with ordered players generalizes both the *Shapley value* (Shapley (1953)) and the Harsanyi set for arbitrary TU-games. In particular, it is equal to the Harsanyi set of the unrestricted game if the digraph is empty, and it is a singleton consisting of the Shapley value if the digraph if complete in the sense that all players dominate each other. As a corollary the Harsanyi set for games with ordered players is always non-empty, whereas the Selectope might be empty.

We provide characterizations of the Harsanyi set and Selectope for games with ordered players using the known properties of efficiency, null player property, disjoint additivity and sign preservation. To characterize the Harsanyi set we add to these properties the inferior player property which reflects the hierarchical dominance by stating that players always earn at least as much as players that are inferior to them. Here player i is inferior to player j if one of the following two conditions holds: (i) i is a subordinate of j, all dividends of coalitions containing j are non-negative and all dividends of coalitions containing i but not j are zero, (ii) i is a superior of j, all dividends of coalitions containing i are non-positive and all dividends of coalitions containing j but not i are zero. It turns out that the Harsanyi set for games with ordered players satisfies these five properties. Moreover, if a solution satisfies these five properties then to every game with ordered players it must assign a set of payoff vectors that is a subset of the Harsanyi set of this game with ordered players. As corollaries we obtain characterizations of the Harsanyi set and Shapley value for unrestricted TU-games.

The Selectope for games with ordered players also satisfies the five properties mentioned in the previous paragraph. (Note that this gives an alternative proof for the Selectope of a game with ordered players being a subset of the corresponding Harsanyi set.) It also satisfies an alternative inferior player property which requires that an inferior player who generates nothing without one of its superiors earns zero in any payoff vector in the Selectope (whenever this solution is non-empty).

After discussing a method for computing the extreme points of the Harsanyi set for any game with ordered players, we apply this solution to *peer group games* as introduced by Branzei, Fragnelli and Tijs (2002). For such games with ordered players the Selectope is a singleton, and both the Shapley value of the unrestricted game and the unique element of the Selectope are extreme points of the Harsanyi set for the game with ordered players. Applying these results to peer group games where the hierarchy has a specific structure, such as a line-graph or a star-graph, we obtain new solutions for, a.o. airport games, sequencing games and auction games.

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