

# Correlated Equilibria in Markov Stopping Games. Numerical Methods and Examples

David M. Ramsey\*and Krzysztof Szajowski†

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## Extended Abstract

### 1 Introduction

Much research has been carried out on various models of stopping games, beginning with the Dynkin game (see Dynkin (1969)). It has often been noted that there may not be a unique equilibrium in such games, if the players are only able to choose their strategies from the set of mixed (randomized) stopping rules (see Szajowski (1994); Neumann et al. (2002)). Hence, communication between the players would be useful in deciding which equilibrium should be played. In such a case, the actions undertaken by the players may be correlated. Thus, the set of possible strategies is extended to the set of correlated stopping times. The idea of correlated strategies was introduced by Aumann (1974).

Little research has been carried out on the role of communication between players in stopping games. Solan (2001) and Solan and Vieille (2002) consider correlated equilibria in general dynamic games. They describe two forms of correlated strategies that may be used in dynamic games. Firstly, they consider stationary correlated strategies, where the strategies of the players are set by communication at the beginning of the game. Such communication may make use of the observation of an external signal (for example, the result of a coin toss). There is no communication at further stages of the game. Secondly, they consider extended correlated strategies, where such a communication process is applied before each decision point of the game.

It is well known (see Gilboa and Zemel (1989)) that a Nash equilibrium is a complicated concept of solution from a computational point of view, whereas the concept of correlated equilibrium is simple. Correlation means a certain limitation of the freedom of the players in selecting their pure actions, because some process of pre-play communication is needed to realize a correlated strategy. However, any player is free to choose any pure action, regardless of the results of the communication process. The aim of this paper is to adopt this approach to nonzero-sum stopping games. An introduction to correlated equilibria in stopping games is given in Ramsey and Szajowski (2004).

The conditions for a pair of correlated stopping times to constitute a correlated equilibrium are given. Various criteria used by the players to correlate their actions are considered. These criteria are based on those used by Greenwald and Hall Greenwald and Hall (2003). Examples using games based on the secretary problem are presented. The notion of a stationary correlated equilibrium is briefly considered.

### 2 Construction of correlated equilibria

The subject of this section is the construction of a correlated equilibrium according to Definition 2 in Ramsey and Szajowski (2004). Each correlated stopping strategy can be presented as

$$(\pi_n^{ss}, \pi_n^{sc}, \pi_n^{cs}, \pi_n^{cc}) = (q_n^1, q_n^2 - q_n^1, q_n^3 - q_n^2, 1 - q_n^3) \quad (1)$$

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\*Instytut Matematyki, Politechniki Wrocławskiej, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, D.Ramsey@im.pwr.wroc.pl

†Instytut Matematyki, Politechniki Wrocławskiej, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, K.Szajowski@im.pwr.wroc.pl

for  $n = 0, 1, \dots, N$ . Denote  $\hat{\mathcal{Q}}_n^N = \{\hat{q} \in \hat{\mathcal{Q}}^N : q_k^1 = q_k^2 = q_k^3 = 0, k = 0, 1, \dots, n-1, q_N^1 = q_N^2 = q_N^3 = 1\}$ . The policy  $\hat{q} \in \hat{\mathcal{Q}}_n^N$  will be denoted  $\hat{q}^{(n)}$ . For a given  $\hat{q}^{(n)} \in \hat{\mathcal{Q}}_n^N$  define

$$\begin{aligned} u_i^{ss}(n, x, \hat{q}^{(n+1)}) &= \mathbf{E}_{X_n} \bar{G}_i(\hat{q}^{(n)}) \mathbb{I}_{\{\lambda^1(\hat{q}^{(n)}) = \lambda^2(\hat{q}^{(n)}) = n\}} \mathbb{I}_{\{X_n = x\}} \\ u_i^{sc}(n, x, \hat{q}^{(n+1)}) &= \mathbf{E}_{X_n} \bar{G}_i(\hat{q}^{(n)}) \mathbb{I}_{\{\lambda^1(\hat{q}^{(n)}) = n, \lambda^2(\hat{q}^{(n)}) > n\}} \mathbb{I}_{\{X_n = x\}} \\ u_i^{cs}(n, x, \hat{q}^{(n+1)}) &= \mathbf{E}_{X_n} \bar{G}_i(\hat{q}^{(n)}) \mathbb{I}_{\{\lambda^1(\hat{q}^{(n)}) > n, \lambda^2(\hat{q}^{(n)}) = n\}} \mathbb{I}_{\{X_n = x\}} \\ u_i^{cc}(n, x, \hat{q}^{(n+1)}) &= \mathbf{E}_{X_n} \bar{G}_i(\hat{q}^{(n)}) \mathbb{I}_{\{\lambda^1(\hat{q}^{(n)}) > n, \lambda^2(\hat{q}^{(n)}) > n\}} \mathbb{I}_{\{X_n = x\}}. \end{aligned} \quad (2)$$

For a given correlated profile, at each stage  $n \in \{0, 1, \dots, N\}$  the players can observe the payoffs in the bimatrix game defined by  $(u_1^{ij}, u_2^{ij})_{i,j \in A}$  defined by (2), where  $A = \{s, c\}$ . Based on the concept of a correlated equilibrium for such a bimatrix game, one can define rational behaviour at stage  $n$ .

**Definition 2** *Let us formulate four different selection criteria for correlated equilibria in a stopping game.*

1. *A utilitarian correlated equilibrium  $\hat{q}_U^* = (\pi_n^{ss}, \pi_n^{sc}, \pi_n^{cs}, \pi_n^{cc})$  is an equilibrium which at each stage  $n = 0, 1, \dots, N$  maximizes the sum of the values of the game to the players*

$$\max_{\hat{q} \in \mathbb{C}\mathbb{E}} \sum_{\gamma \delta \in B} \sum_{i=1}^2 \pi_n^{\gamma\delta} u_i^{\gamma\delta}(n, x, \hat{q}) = \sum_{\gamma \delta \in B} \sum_{i=1}^2 \pi_n^{\gamma\delta} u_i^{\gamma\delta}(n, x, \hat{q}_U^*) \text{ on } \{\omega : X_n = x\}, \quad (3)$$

where  $B = \{ss, sc, cs, cc\}$ .

2. *An egalitarian correlated equilibrium  $\hat{q}_E^*$  is an equilibrium  $\hat{q} = (\pi_n^{ss}, \pi_n^{sc}, \pi_n^{cs}, \pi_n^{cc}) \in \mathbb{C}\mathbb{E}$ , which at each stage  $n = 0, 1, \dots, N$  maximizes the minimum value*

$$\max_{\hat{q} \in \mathbb{C}\mathbb{E}} \sum_{\gamma \delta \in B} \min_{i \in \{1, 2\}} \pi_n^{\gamma\delta} u_i^{\gamma\delta}(n, x, \hat{q}) = \sum_{\gamma \delta \in B} \min_{i \in \{1, 2\}} \pi_n^{\gamma\delta} u_i^{\gamma\delta}(n, x, \hat{q}_E^*) \text{ on } \{\omega : X_n = x\}. \quad (4)$$

3. *A republican correlated equilibrium  $\hat{q}_R^*$  is an equilibrium which at each stage  $n = 0, 1, \dots, N$  maximizes the maximum value*

$$\max_{\hat{q} \in \mathbb{C}\mathbb{E}} \sum_{\gamma \delta \in B} \max_{i \in \{1, 2\}} \pi_n^{\gamma\delta} u_i^{\gamma\delta}(n, x, \hat{q}) = \sum_{\gamma \delta \in B} \max_{i \in \{1, 2\}} \pi_n^{\gamma\delta} u_i^{\gamma\delta}(n, x, \hat{q}_R^*) \text{ on } \{\omega : X_n = x\}. \quad (5)$$

4. *A libertarian  $i$  correlated equilibrium  $\hat{q}_L^*$  is an equilibrium which at each stage  $n = 0, 1, \dots, N$  maximizes the value of the game to Player  $i$ .*

$$\max_{\hat{q} \in \mathbb{C}\mathbb{E}} \sum_{\gamma \delta \in B} \pi_n^{\gamma\delta} u_i^{\gamma\delta}(n, x, \hat{q}) = \sum_{\gamma \delta \in B} \pi_n^{\gamma\delta} u_i^{\gamma\delta}(n, x, \hat{q}_L^*) \text{ on } \{\omega : X_n = x\}. \quad (6)$$

Using criteria given in the definition 2, the appropriate correlated equilibria can be defined by recursively solving a set of linear programming problems. In the case of libertarian and utilitarian equilibria, the linear objective functions are given by equation (6) and (3), respectively. The constraints result from the definition of a correlated profile. Since, the feasible set of this linear programming problem is non-empty (a correlated rational strategy always exists), such solutions always exist. In the case of the republican equilibrium, at each stage of the recursion we can solve the following two linear programming problems:

1. The maximization of the value of the game to Player 1 subject to the constraints from the definition of correlated equilibrium, together with the constraint that the value of the game to Player 1 is at least the value of the game to Player 2.
2. The maximization of the value of the game to Player 2 subject to the constraints from the definition of correlated, together with the constraint that the value of the game to Player 2 is at least the value of the game to Player 1.

In order to find a correlated strategy satisfying the republican criterion, it suffices to choose an appropriate solution from the solutions to these two problems (maximizing the maximum value). It should be noted that the union of the feasible sets of these two linear programming problems is the set of correlated rational strategies. Hence, at least one of these feasible sets must be non-empty. In the case where only one feasible set is non-empty, the solution of the corresponding linear programming problem maximizes the maximum value.

An analogical procedure using two linear programming problems can be used to find a correlated equilibrium satisfying the egalitarian criterion. When the objective function is defined by the maximization of the value of the game to Player  $i$ , then the additional constraint in the linear programming problem is that the value of the game to Player  $i$  is not greater than the value of the game to the other Player.

It should be noted that although the existence of a correlated equilibria of the types given above is guaranteed, it is possible that such equilibria are not unique.

### 3 Correlated Equilibria in a Game Based on the Best Choice Problem

Games related to the best choice problem considered in Szajowski (1994) are used as illustrations of the method of finding solutions. Two players observe a sequence of  $N$  objects presented in a random order. The  $n$ -th object is observed at moment  $n$ , ( $1 \leq n \leq N$ ). The objective of each player is to obtain the most valuable object. They can only observe the relative rank of an object with respect to the objects already seen. If the  $n$ -th object is the  $k$ -th best seen so far, then its relative rank  $R_n$  is equal to  $k$ . At each moment  $n$  both players have two possible actions, reject the object and continue inspecting (denoted  $c$ ) and accept the object (denoted  $s$ ). Each player can obtain at most one object and on obtaining an object ceases searching. If just one player wishes to accept an object, then he obtains that object. If both players wish to obtain an object, then Player 1 obtains it with probability  $\alpha$ ,  $\alpha \in (0, 1)$ , otherwise Player 2 obtains it. The player not obtaining the object is free to carry on inspecting the sequence.

A solution of each type defined in the previous section is constructed.

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