Serial cost sharing

Elena Yanovskaya *

St.Peterburg Institute for Economics and Mathematics,
Russian Academy of Sciences,
Tchaikovsky st. 1, 191187 St.Peterburg, RUSSIA

Cost sharing methods with variable demands of heterogeneous goods are considered. One of the well-known methods is the serial one which verifies the *serial property*: an agent's cost share does not depend on sizes of demands larger than his own. The axiomatization of the serial method due to Friedman and Moulin [1] uses the upper bound property for the problems with *homogenous* cost functions. Such a "restricted domain" axiom is not compelling because it is normatively meaningful only on this restricted domain – the homogeneous cost sharing problems.

Other popular cost sharing methods: the Shapley-Shubik and the Aumann-Shapley methods satisfy ordinality axiom which imposes invariance of the cost shares under arbitrary increasing transformations of the measuring scales. The serial method does not verify ordinality, but it verifies its weaker version *coordinality*: independence of the cost shares under *common* ordinal transformations of the agents utilities. In this report an axiomatic characterization of the serial method is given using coordinality as one of the axioms.

A *heterogenous cost sharing problem* is a triple $\langle N, C; q\rangle$, where N is a finite set of agents, $C : \mathbb{R}^N_+ \to \mathbb{R}_+$ is a nondecreasing cost function such that $C(0) = 0$, and $q \in \mathbb{R}^N_+$ is a profile of demands. When the set N is fixed we shall denote the problem $\langle N, C, q\rangle$ simply by (C, q).

A *solution* of the problem $\langle N, C, q\rangle$ is a vector $y \in \mathbb{R}^N_+$ such that $\sum\limits_{i\in N} y_i = C(q)$.

A *cost sharing method (rule)* is a mapping φ associating with any problem $\langle N, C, q\rangle$ a solution $y = \varphi(N, C, q)$. The cost shares y_i should reflect responsibilities in generating the costs. A minimal requirement to that effect is that an agent who is not generating any cost should pay nothing. The Dummy axiom conveys just that idea.

Dummy (DUM). If the function C does not depend on a variable $i \in N :\ C(q) = C(q\|_i 0)$, then $y_i = \varphi(N, C, q) = 0$ for every $q \in \mathbb{R}^N_+$.

The additivity axiom is formulated as usual:

Additivity (ADD).

$$\varphi(N, C^1 + C^2), q) = \varphi(N, C^1, q) + \varphi(N, C^2, q) \text{ for all } N, C, q.$$

*Research supported by the Russian Foundation for Basic Research, grant 04-06-80316

The set of all methods satisfying Dummy and Additivity is denoted by $\mathfrak{F}$*(DUM,ADD)*. A similar notation will be used for cost sharing methods satisfying other properties as well.

Symmetry (SYM). A cost sharing method φ is *symmetric* if for any problem $\langle N, C, q\rangle$ with a symmetric in the demands $i, j \in N$ cost function C the equaility $q_i = q_j$ implies $\varphi_i(N, C, q) = \varphi_j(N, C, q)$.

A *path* γ to a point $q \in \mathbb{R}^N_+$ is a continuous nondecreasing curve connecting the origin $\mathbf{0}$ with q. Hence, there exists a parametric representation $\gamma(t, q), t \in [0, 1]$ for every path γ :

$$x \in \gamma \iff x_i = \gamma_i(t, q) \text{ for all } i \in N \text{ and some } t \in [0, 1]. \tag{1}$$

In representation (1) the functions $\gamma_i(q, t)$ are continuous and nondecreasing in t. ¿From the representation it follows that $\gamma(q, 0) = 0, \gamma(q, 1) = q$ for all $q \in \mathbb{R}^N_+$.

Denote by $\mathfrak{C}$ the set of all continuously differentiable cost functions. With any path γ we can associate the cost sharing method x^γ such that for any $C \in \mathfrak{C}$

$$x^\gamma_i(C, q) = \int_0^1 \partial_i C(\gamma(q, t)) d\gamma_i(q, t), \tag{2}$$

where $\partial_i C$ denotes the i-th partial derivative of C.

Haimanko [2] proved that such *path generated* methods are the only methods which are the extreme points of the set $\mathfrak{F}(ADD, DUM)$.

Sprumont [3] introduced a property of the cost sharing methods known as ordinality. To define it we give some notation.

Let N be fixed. Consider a list $(f_1, \dots, f_n)$, of n bijections from $\mathbb{R}_+$ onto itself. For each cost function $C : \mathbb{R}^N_+ \to \mathbb{R}_+$ define another cost function $C^f : \mathbb{R}^N_+ \to \mathbb{R}_+$ by

$$C^f(q) = C(f(q)), \text{ where } f(q) = (f_1(q_1), \dots, f_n(q_n)).$$

We call the problem $\langle C^f, f^{-1}(q)\rangle$ the *ordinal transformation* of the problem $\langle C; q\rangle$. Here $f^{-1}(q) = (f_1^{-1}(q_1), \dots, f_n^{-1}(q_n))$.

When we consider a special class of cost functions (e.g. continuous, differentiable etc.) we suppose that the functions f_i possess the same properties in order the ordinal transformations belong to the same class.

Two problems $\langle C, q\rangle$ and $\langle C', q'\rangle$ are *ordinally equivalent* if there exists an ordinal transformation f such that

$$C' = C^f \quad \text{and} \quad q = f(q').$$

Ordinality (ORD). Let N be an arbitrary fixed set. If $\langle N, C; q\rangle$ and $\langle N, C'; q'\rangle$ are two ordinally equivalent problems, then $\varphi(N, C, q) = \varphi(N, C', q')$.

With every cost sharing problem $\langle N, C, q\rangle$ a TU game $\langle N, v\rangle$ can be associated such that $v(S) = C(q_S, \mathbf{0}_{N\setminus S})$. Then every single-valued cooperative game solution can be considered as a cost sharing method. Evidently, all such methods verify ordinality. Thus, Sprumont [3] has characterized the Shapley-Shubik method as a unique method from $\mathfrak{F}(ADD, DUM, SYM, ORD)$.

The *serial cost sharing method* s is given by the following formula:

$$s_i(C,q) = \int_0^{q_i} \partial_i C((te) \wedge q)dt, \tag{3}$$

where $e = (1,\ldots,1)$, $(a \wedge b) = \min\{a,b\}$.

Definition (3) implies that $s \in \mathfrak{F}(ADD, DUM)$ and it is a path generated method. However, it does not satisfy ordinality. It satisfies only some its weakening, if instead of n bijections $(f_1,\ldots,f_n)$ in the definition of ordinal transformation we consider the same function $f = f_i, i = 1,\ldots,n$. Such a transformation of a cost sharing problem we call the *coordinal* transformation. The coordinality property of a cost sharing method φ is formulated as follows:

Coordinality (CORD). Let N be an arbitrary fixed set. If $\langle C; q\rangle$ and $\langle C^f; f^{-1}(q)\rangle$ is a coordinal transformation of $\langle C; q\rangle$ then $\varphi(C,q) = \varphi(C^f, f^{-1}(q))$.

The problems $\langle C; q\rangle$, $\langle C'; q'\rangle$ which are coordinally equivalent keep the order between the numbers $q_i, i \in N$, i.e. the correspondence "poor–rich" that is essential in the serial method. Evidently, ORD implies CORD.

The last property of cost sharing methods which is necessary in the sequel is

Continuity (CON). A cost sharing method φ is *continuous* if the function $\varphi(N,C,q)$ is continuous in C and q.

Lemma 1 *If $\varphi(N,C,q)$ is an extreme point of the set $\mathfrak{F}(ADD, DUM)$, and satisfies SYM, CON, and CORD, then it is the serial method.*

Lemma 1 is not yet an axiomatic characterization of the serial method, since it only singles out this method from the set of extreme points of $\mathfrak{F}(ADD, DUM)$. Therefore, we should add another axiom which, together with Additivity and Dummy, will characterize the property "to be an extreme point" of the set $\mathfrak{F}(ADD, DUM)$.

Decomposition (DECO). A method φ satisfies the *decomposition* property if for every problem $\langle N, C; q\rangle$ for each $i \in N$ and $z_i \in (0, q_i)$ there exist $z_j \in [0, q_j], j \in N \setminus \{i\}$ such that for any $k \in N$

$$\varphi_k(N,C,q) = \varphi_k(N,C,z) + \varphi_k(N,C^z, q-z), \tag{4}$$

where $z = (z_i)_{i\in N}$, $C^z(x) = C(z+x)$, $x \in [0, q-z]$.

For extreme points of the set $\mathfrak{F}(ADD, DUM)$ the decomposition property may be treated as a path independence property. In fact, if a method φ is path generated, then the agent's payoff is its average marginal gain along the corresponding path. Path independence of such a method means that in each point of the path we could give the payoff accumulated to this point to each agent and then put them to the new problem, beginning from that point and finishing at the end of the initial problem. Since the sums of payoffs in two problems are equal to the corresponding agents' payoffs in the initial problem, then, being in any point of the path the further gains of agents only depend on the next piece of the path, not on the way coming to the point.

The following theorem characterizes the serial method with the help of the decomposition property:

Theorem 1 *The serial method is the only method satisfying ADD, DUM, SYM, CORD, CONT and DECO.*

References

[1] Friedman E., Moulin H. Three methods to share joint costs or surplus. J. Econ. Theory, **87**, (1999), 275-312.

[2] Haimanko O., Partially symmetric values, mimeo, Hebrew University, Jerusalem, 1998.

[3] Sprumont Y., Ordinal cost sharing.. J. Econ. Theory, **81**, (1998), 126-162.