The existence of globally stable price mechanism for pure exchange models with upper semicontinuous multivalued excess demand

Aleksandra Arkit
Faculty of Mathematics, Computer Sciences and Econometrics
University of Zielona Góra, ul. Szafrana 4a, 65-516 Zielona Góra

Introduction The aim of the talk is to present sufficient conditions for the upper semicontinuous multivalued excess demand, guaranteeing the existence of some globally stable price mechanism. We consider two different price mechanisms: sign–compatible and α–compatible with the excess demand. Our conditions depend on values of the excess demand and the location of the corresponding price system with respect to the equilibrium price system. We show that there exist adequate price mechanisms in Scarf's example (when the excess demand is singlevalued) and in examples with upper semicontinuous multivalued excess demand.

Model Consider a model of pure exchange with the multivalued excess demand $E : \overline{R}^n_+ \setminus \{\mathbf{0}\} \rightsquigarrow R^n$ (where $\overline{R}_+ = [0, +\infty)$) satisfying the following natural hypothesis:

(a0) E has nonempty, closed and convex values

(a1) E is upper semicontinuous

(a2) E satisfies Walras' Law: $\left\langle u, p \right\rangle = 0$ for all $u \in E(p)$ (where $\langle \cdot, \cdot \rangle$ denotes the inner product)

(a3) E is positive homogeneous of degree zero

(a4) E satisfies boundary condition: if $p_i = 0$ then $u_i \geq 0$ for all $u \in E(p)$.

The excess demand is the difference between the total demand and the total supply of commodities, which are exchanged on the market. We assume that this map depends only on commodity bundle's price vector. The hypothesis (a0)–(a4) guarantee the existence of at least one Walras equilibrium, i.e. a point p^*, such that $\mathbf{0} \in E(p^*)$ (compare Debreu (1984)). Following Samuelson

(1941) we assume that the path of prices, which starts at fixed p^0, is a solution of the differential equation

$$\frac{dp}{dt} = g(p), \qquad p(0) = p^0. \tag{1}$$

The continuous function $g : \overline{R^n_+} \setminus \{\mathbf{0}\} \to R^n$, on the right hand side of the Eq.(1), which satisfies (when substituted for E) (a2)–(a4) and the condition: $g(p^*) = \mathbf{0}$ if and only if $p^* \in \mathcal{P}_E = \{p \in R^n_+ : \ \mathbf{0} \in E(p)\}$, we will call a *price mechanism*. We say that a price mechanism is *globally asymptotically stable* if any price trajectory $p(t)$, which is a solution of (1) for any initial point p^0, satisfies condition

$$\forall \varepsilon > 0 \, \exists \, \delta > 0 \, \forall t \geq 0 \ |p^0 - p^*| < \delta \ \Rightarrow \ |p(t) - p^*| < \varepsilon, \tag{2}$$

and it converges to some $p^* \in \mathcal{P}_E$, when t tends to infinity.

Let us recall that every price trajectory for the price mechanism g is located on the nonnegative part of the sphere $\overline{S_+}(|p^0|) = \{p \in \overline{R^n_+} : \ |p| = |p^0|\}$ (because of (a2)). Since g satisfies (a3) we can treat such price adjustment process as a continuous tangent vector field on $\overline{S_+} = \{p \in \overline{R^n_+} : \ |p| = 1\}$. This is the reason why we can restrict a domain of price mechanisms to $\overline{S_+}$.

Problem One can consider different kinds of price mechanisms. Let F be a given multivalued map from $\overline{S_+}$ into $2^{R^n} \setminus \{\emptyset\}$. We say that a price mechanism g is specified by F if $g(p) \in F(p)$ for all $p \in \overline{S_+}$. Since g has to have zeros at equilibrium points p^* we impose on F the following condition: $F(p^*) = \{\mathbf{0}\}$ if and only if $\mathbf{0} \in E(p^*)$. We are going to give sufficient conditions for the excess demand, guaranteeing the existence of some globally asymptotically stable price mechanism g. In other words, we ask when there exists the continuous selection g of the multivalued map $p \longmapsto F(p) \cap H_0(p)$ (where $H_0(p) = \{u \in R^n : \langle u, p \rangle = 0\}$), such that any trajectory of an autonomous equation (1), with $p^0 \in S_+$, is convergent to some equilibrium point $p^* \in \mathcal{P}_E$ and satisfies (2).

A price mechanism g describes a 'classical' price adjustment process if it is compatible with the excess demand in the following sense. If at some price system p i–th coordinate of all $u \in E(p)$ is positive then $g_i(p) \geq 0$ If i–th coordinate of all $u \in E(p)$ is negative then $g_i(p) \leq 0$. If there exists $u \in E(p)$ such that its i–th coordinate is zero then $g_i(p) = 0$. Thus g is specified by F

defined by $F(p) =$

$$= C^{\uparrow}(E(p)) = \bigcap_{v \in E(p)} \{u \in R^n : \text{ if } v_i \neq 0 \text{ then } u_i v_i \geq 0, \text{ if } v_i = 0 \text{ then } u_i = 0\}.$$

We call such price mechanism g *a sign–compatible* with the excess demand.

When the price mechanism is sign–compatible with the excess demand we know that a price change vector and all excess demand vectors from $E(p)$ have to be in the same orthant of R^n. In special cases, it permits the situation when these directions are relatively divergent (the angle between these vectors could be nearly $\frac{\pi}{2}$). This motivates considering a price mechanism g where $g(p)$ forms with all excess demand vectors from $E(p)$ an acute angle and not necessarily $g(p)$ has to have the same signs as all excess demands from $E(p)$ (see Arkit (2003)). Let $\alpha \in \left[0, \frac{\pi}{2}\right]$. We say that a price mechanism g is *α–compatible* with the excess demand E if $g(p)$ forms with all vectors $u \in E(p)$ an angle less than or equal to α. Thus g is specified by F defined by

$$F(p) = C^{\alpha}(E(p)) = \bigcap_{v \in E(p)} \{u \in R^n : \left\langle u, v \right\rangle \geq |u||v| \cos \alpha\}.$$

Examples Let us consider the pure exchange model with three commodities and three agents. Let us assume that the agents have got following initial endowements: $w^1 = (1,0,0)$, $w^2 = (0,1,0)$ and $w^3 = (0,0,1)$. The excess demand is defined from the utilities of agents, which are: in *Example 1* (Scarf's example): $u^1(x) = \min\{x_1, x_2\}$, $u^2(x) = \min\{x_2, x_3\}$, $u^3(x) = \min\{x_1, x_3\}$; in *Example 2*: $u^1(x) = \max\left\{\frac{1}{2}x_2, \frac{1}{2}x_3\right\} + \min\{x_2, x_3\}$, $u^2(x) = \min\{x_1, x_3\}$, $u^3(x) = \min\{x_1, x_2\}$. We show that in both examples there exist price mechanisms sign– compatible as well as α–compatible with the excess demand, which are globally asymptotically stable.

Arkit A., (2003), *Globally stable price dynamics*, *Journal of Mathematical Economics* **39**, 27–38

Debreu G., (1984) *Existence of Competitive Equilibrium* in M.D. Intriligator, K.J. Arrow ed., *Handbook of Mathemtical Economics* Vol.II, *North–Holland: Amsterdam, New York, Oxford*

Samuelson P.A., (1941), *The stability of equilibrium: comparative statics and dynamics*, *Econometrica* **9** No. 2, 97–120