Stable Effectivity Sheaves

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We present a model of interaction that imbeds aspects of both coalitional and strategic theory. The primitives of this model are a finite set of players $N = \{1, \ldots, n\}$ and a set A of outcomes (also called positions, states, proposals, alternatives) that can be either finite or more generally a topological space. When a position a is achieved some "patterns" appear to be within the reach of the players. A pattern is a n- tuple $(B_1, \ldots, B_n)$ where each B_i is a subset of A with the condition that one at least is not empty. Let $\mathcal{E}(a)$ be the set of such patterns. An effectivity structure is a collection $(\mathcal{E}(a), a \in A)$. The interpretation of the relation $(B_1, \ldots, B_n) \in \mathcal{E}(a)$ is that whatever are the means that achieve position a then some qualified coalition S is able to reach some position in $\cap_{i \in S} B_i$, thus potentially destabilising the initial position a. Assume that a profile of preferences is given, then such a move is likely to be undertaken if the positions in B_i are preferred to a by player i. We do not describe the "strategic" background which is behind the implementation of the initial position or the moves undertaken to depart from it, even though given such a strategic background - a game form and an equilibrium concept in our setting - one can construct such a model. With this respect our model extends coalitional notions like effectivity functions. On the other hand the set of patterns may embed enough interactive description to represent those interactions derived from strategic moves in an explicit strategic setting.

Thus we may compare our model on one hand with effectivity functions, on the other with game forms together with a solution principle. An effec-

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tivity function is described by a collection $(E(S), S \in \mathcal{S})$ where $\mathcal{S}$ is a set of qualified coalitions, and for each $S \in \mathcal{S}$ a set $E(S)$ of non-empty subsets of A. The relation $B \in E(S)$ is interpreted as follows: whatever is the present position a and whatever are the means by which a is achieved, coalition S can put the final outcome in B. Just as in our model, neither the actions that achieve a nor the moves to reach B are explicit. We may reflect the effectivity function E in our model by defining $(B_1, \ldots, B_n) \in \mathcal{E}(a)$ if and only if for some coalition $S \in \mathcal{S}$ and some $B \in E(S)$, $B_i = B$ for all $i \in S$. In particular $\mathcal{E}(a)$ does not depend on a.

On the strategic side let $g : X_1 \times \cdots \times X_n \to A$ be a game form and let $\mathcal{S}$ be a set of qualified coalitions. When a position a is achieved say via $x = (x_1, \ldots, x_n)$, $g(x) = a$, then some coaltion may try to upset a by deviating. We can define those patterns $(B_1, \ldots, B_n)$ such that for any scenario x leading to a some qualified coalition S can move in order to drive the outcome into $\cap_{i \in S} B_i$. We thus have associated to the game form an effectivity structure that reflects the interactive power induced by $\mathcal{S}$.

In our model a position a is undominated for a given preference profile if no pattern $(B_1, \ldots, B_n) \in \mathcal{E}(a)$ is preferred to a by all players. We shall call stable an effectivity structure that possesses an undominated position for any preference of some class. When applied to the effectivity structure associated to a strategic game form and a set of qualified coalitions $\mathcal{S}$ then an undominated position is just an $\mathcal{S}$ - equilibrium outcome. Thus studying stability provides information about solvability of a game form. The remarkable gain in this passage is that solvability, for any qualified class of coalitions $\mathcal{S}$, can be studied under the same type of model.

Effectivity structures and their stability for a finite set A has been presented in [**?**]. When A is a topological space it is more elegant to start by effectivity sheaves or presheaves. An effectivity presheave is a collection $(\mathcal{H}(U))$ where U is any open set. Each $\mathcal{H}(U)$ is a set of patterns. The interpretation of the relation $(B_1, \ldots, B_n) \in \mathcal{H}(U)$ is that whatever are the means by which a position is achieved in U then some qualified coalition S is able to reach some position in $\cap_{i \in S} B_i$. We prove that when A is compact Haudorff stability of closed valued effectivity sheaves is equivalent to the absence of cycles. As an application we have a necessary and sufficient condition for $\mathcal{S}$-solvabilty of a continuous game form.

References

[1] Abdou J., Keiding H., 2003. *On necessary and sufficient conditions for solvability of game forms*, Mathematical Social Sciences 46, 243-260.