

# Jerzy Łoś

## 1920-1998

### 1. Elements of biography

Jerzy Łoś was born on March 22, 1920 in Lvov, in the very same year that saw the emergence of the Polish School of Mathematics through its new and novel journal “Fundamenta Mathematicae”. Many years later “Fundamenta Mathematicae” would host numerous important mathematical results of Professor Łoś.

In the short period of peace preceding World War II Łoś had had the time to complete his secondary education and enter the Jan Kazimierz University in Lvov in 1937, one year after his father’s death. The war interrupted his studies as well as his usual course of life. To earn his living, he took up a clerk’s job at the sugar plant Lublin (1942-43) and then became a bookkeeper at the real estate Tarnogóra (1943-44).

The end of the war was marked by his marriage in 1945, of which he was to have two sons, and the renewal of his university studies at the Maria Curie-Skłodowska University in Lublin; in his curriculum mathematics was now complemented by philosophy. In 1947 Łoś got his master degree in philosophy and – having moved to Wrocław – became an assistant first at the Chair of Physics, and later at the Chair of Logic at the Wrocław University. Here he made his PhD degree in sciences in 1949 (“On logical matrices”[4]), and six years later, this time at the Mathematical Institute of the Polish Academy of Science, the second scientific degree corresponding to the “habilitacja” (based on two papers: “The algebraic treatment of the methodology of elementary deductive systems” [18] and “On the extending of models” [20]). In the meantime he had moved to the University of Toruń, where he worked until 1961, receiving the nominations to the state titles of extraordinary professor in 1954 and ordinary (“full”) professor in 1957.

Łoś joined the Mathematical Institute of the PAS in 1949 and remained there till his retirement in 1991. At first in the Group of Real Functions, in 1952 he became head of the Group of Algebra and in the years to follow greatly contributed to the establishment of a strong algebraic centre. He left this position at the beginning of 1968 only to concentrate on the new task of deputy head of the Institute for scientific affairs. After one year he added to this the responsibilities of head of the Mathematical Methods in Economics section. In 1972 he moved to the Institute of Computer Science of the PAS, where he continued to head the same section.

Author of one of the most efficient tools in foundations of mathematics, the method of ultraproducts based on the widely known and used Łoś’s theorem on ultraproducts, his work was duly esteemed both in

Poland and abroad. As a visiting professor in 1959/60 he presided a model-theoretic seminar at the Berkeley University together with Alfred Tarski, in 1962/63, again in his quality of visiting professor, he ran a joint seminar with Bjarni Jónsson on universal algebra at the same university. The students at the University of Aarhus in Denmark had the chance to follow his lectures on mathematical methods in economics in 1967, in 1969 he gave a series of lectures on the theory of economic models at the Poincaré Institute in Paris. Similar lectures took place during his long stays in Yale (1973) and the University of Wisconsin at Madison (1978/9). Always very active in all the fields of professional activity, he participated in many important international mathematical events, including International Congresses of Logic, Methodology and Philosophy of Sciences and International Congresses of Mathematicians. In 1979 he was elected President of the International Union of History and Philosophy of Sciences, Division of Logic, Methodology and Philosophy of Sciences and held this position till 1983, becoming Past President until 1988. The University of Hagen in Germany endowed him with the title of *doctor honoris causa* in 1995.

All his professional career was accompanied by intense activities addressed to the mathematical community. Together with Stanisław Jaśkowski he established the Toruń branch of the Polish Mathematical Society and initiated the now traditional yearly Marcinkiewicz prize for the best paper presented by a young mathematician. He pioneered research in algebra, not only acting as a teacher to a group of talented young people in Toruń and Warsaw, but also by organizing national and international algebraic conferences in Poland. Equally novel was the creation of a research centre on mathematical methods in economics at the Mathematical Institute of the PAS. Indeed, it is quite unusual for one person to initiate and organize research groups in more than one domain of mathematics as Jerzy Łoś did.

Professor Łoś profoundly influenced his students, colleagues and all those who had the chance to meet him. His enthusiasm for scientific work turned out to be contagious, as can be seen with his numerous followers, many of them professors in mathematics or economics themselves by now. A broad-minded man, he was known by his independent and courageous standpoints, which brought him to actively support the “Solidarity” movement in the early 1980’s; his opinions and views were highly appreciated by many of those who met him, even if they differed from their own.

The main domains of Łoś’s research activities are presented and commented below. They have already been highly evaluated by his contemporaries, who bestowed several scientific awards on him, and by the Polish Academy of Science, which elected him its member as early as 1964 (member-correspondent) and confirmed it in 1983, making him a full member of the Academy.

In the spring of 1996 Jerzy Łoś suffered a severe brain stroke. After two years of serious illness he died on June 1, 1998.

## Mathematics and its foundations

The first papers of Łoś fall into the area of foundations of mathematics. His paper "Effectiveness of the representation theory for Boolean algebras" [15] brings a link with algebra. It contains a proof of effective equivalence of Stone's theorem on the existence of prime ideals in Boolean algebras and the theorem stating that the product of a family of compact non-empty topological spaces is compact and non-empty. This result, belonging to foundations of mathematics, became for Łoś a point of departure for various applications in the theory of groups (mainly abelian), model theory and universal algebra.

In the theory of abelian groups Łoś initiated a new trend by applying topological and set-theory methods. He developed a theory of complete direct sums of infinite cyclic groups and characterized groups that are direct summands in any group containing them as pure subgroups. He introduced several very important new concepts in the theory of abelian groups such as the famous slender groups, hereditary set of generators, generalized limits in algebraically compact groups etc. His papers in this domain stimulated numerous works not only by his students, but also by foreign algebraists as well.

Łoś showed that there exist numerous possibilities of deducing "infinite from finite" in cases when some mathematical objects are not equipped with compact topology but only admit some sufficiently natural compact topology. The first of those applications concerned the ordering relations in groups but the most important and best known ones refer to a class of abelian groups that are direct summands in every group in which they are pure. Łoś proved that groups in this class coincide with direct summands of groups which admit a compact topology. This class was shown to be the class of Kaplansky's algebraically compact groups.

Łoś's construction of a pure embedding of an arbitrary abelian group into a product of generalized cyclic  $p$ -groups was later used and extended in relative homological algebra.

A quite distinct link between foundations of mathematics and the theory of abelian groups was exposed by Łoś in the theory of products of abelian groups, by showing its connections with general problem of measure in set theory (independently of Zeeman). For some groups constructed from products he obtained and suggested structural results related to algebraic compactness. In this connection his results on infinite systems of linear equations in abelian groups should also be mentioned.

Some early fundamental papers of Łoś on foundations of mathematics, in particular model theory, were related to Birkhoff's theory of universal algebras. The first papers on algebras with infinitely many infinitary operations were written under his influence. These papers together with some other by Łoś have

made a very essential impact on the development of universal algebra, in particular on the theory of equationally definable classes of algebras.

The papers [14] and [19] belong to the very classic core of model theory and are quoted in every handbook of the subject. Both gave strong stimulus to very fruitful investigations grasping the very central problems of foundations of mathematics in those days.

In [14] Łoś stated what is now known as the Łoś-Vaught test: *Suppose that a consistent countable theory  $T$  has only infinite models and  $T$  is  $\alpha$ -categorical for some infinite cardinal  $\alpha$ . Then  $T$  is complete.* He also formulated there his famous categoricity conjecture: *A countable theory categorical in one uncountable power is categorical in any uncountable power.*

The intuition behind must have been of a neatly algebraic origin, coming from the transcendence base phenomenon in vector spaces or algebraically closed fields. Łoś's conjecture inspired some of the best works on the subject. In particular, partial results were obtained by Vaught and the general proof is given in the brilliant work by Morley *Categoricity in power*, Trans. Amer. Math. Soc. 114 (1965), 514-538.

The paper [19] is always quoted when the ultraproduct construction is mentioned. The very idea of the construction goes back to the work of Skolem (1934) who constructed a non-standard model of arithmetic. In [19] the general reduced product construction is introduced and the fundamental theorem relating the theory of an ultraproduct to the theories of its factors is (implicitly) stated. The power of the ultraproduct method is commonly known; its elegance can be well appreciated in Kochen's *Ultraproducts in the Theory of Models*, Ann. Math. 2, 74 (1961), 221-261.

In a series of papers [20, 21, 24, 36, 43, 47] the authors consider so called preservation problems (some of the results had been anticipated in [19]).

In [20] Łoś proved that a theory  $T$  is preserved under extensions of models if and only if  $T$  has a set of existential axioms (cf. also A. Robinson *On a problem of Henkin*, JSL 21 (1956), 33-35).

In [24] a family  $M$  of models of a theory  $T$  is considered and a necessary and sufficient condition for the existence of a model of  $T$  which is the common extension of all the models in  $M$  is given.

In [21] and [36] the authors consider the unions of increasing sequences of models. A theory  $T$  is preserved under such unions if and only if it has a set of universal-existential axioms (cf. also Chang *On unions of chains of models*, Proc. Amer. Math. Soc. 10 (1959), 120-127).

In [43] the authors give some rather simple necessary and sufficient condition for a family of homomorphisms (congruences) to be productable. (Given a family of homomorphisms of models  $h_t: A \rightarrow A_t$ ,  $t \in T$ , we say that it is productable if the map  $h: A \rightarrow \prod A_t$  such that  $h(a)(t) = h_t(a)$  is a homomorphism.)

The paper [47] sharpens some results of [21] by giving a criterion for the existence of a common extension for two models of a set of universal axioms.

There are also some unpublished results of Łoś concerning the connections between logic and probability theory, which probably marked the beginning of his interest in the applications of mathematics.

A characteristic feature of his entire research activity is his ability to find very natural links between problems and methods of various domains of mathematics. This tendency can also be clearly seen in his later works on mathematical methods in economy.

### **Applications of mathematics**

A shift in the interest of Jerzy Łoś from theoretical mathematics towards its applications coincides with his moving from Toruń to Warsaw in 1961. Inspirations may be derived from his acquaintance with Hugo Steinhaus, mentioned by him at various occasions. Undoubtedly his previous works on foundations of probability theory and his research in algebra should be considered direct reference spots to be traced back in Łoś's own activities.

His research in the area of mathematical economics was mainly focused on two subjects:

- production processes, in particular the theory of von Neumann models and
- dynamic decision processes, in particular optimal inventory problems.

The research on von Neumann models (which are production models, described in their basic form by a pair of matrices of the same dimension, comprising data of the production processes possible in a given economy) are probably the most important achievements of Jerzy Łoś in the area of mathematical economics. The results of Jerzy Łoś, often obtained jointly with his students and collaborators, dealt both with properties of the original von Neumann models themselves (especially questions related to equilibria) as well as their generalisations and applications. In particular, Łoś investigated the relations of von Neumann models to game theory and generalisations of models to the case where commodity bundles are ordered not in the usual way, but rather by means of a cone in a Euclidean space. This research also required fundamental studies of the theory of such cones. Applications of von Neumann models studied by Łoś include constructions describing by means of those models the economic concepts of labour, wages and consumption along with export-import processes within so called open models. In several papers he compared Walras equilibria with those in von Neumann models and generalised Samuelson's nonsubstitutability theorem within the framework of von Neumann models. He also attempted to present a comprehensive theory of von Neumann models and, more generally, of linear economic models; these results are only available in research reports. The von Neumann models have their established place in the

economic literature (even though nowadays non-linear models are more frequently used). This is also due to the contributions of Jerzy Łoś.

Another stream of his scientific interest included the theory of optimal storage of inventories and, more generally, the theory of dynamic decision processes, both with discrete and continuous time. Łoś formulated and studied one of the first, definitions of horizon in dynamic programming (horizon is a time period such that decisions based only on data concerning this period can be continued to form an optimal, possibly infinite program) and his definition is still used in the literature. The research of Jerzy Łoś in graph theory is also related to optimal inventory problems (vertices of a graph are then usually interpreted as time moments at which decisions concerning the state of inventory can be made); some results deal with generalisations of König's theorem.

In probability theory Łoś's scientific research was mainly focused on its theoretical foundations; the roots of his ideas can be traced back to his earlier works in logic – probability was then interpreted in terms of functions on formulas. His research interest in the epidemic theory of Jerzy Neyman brought a closer link with applications.

Among his publications there are also several expository papers on applications of mathematics in general; some of them investigate the concept of „applications of mathematics” itself.

Professor Łoś was an open minded scientist, with a keen eye on various applications of mathematics in different areas of everyday life, an excellent presenter and advocate of applications of mathematics and a teacher stimulating others towards independent research. He originated in Poland a new stream of research in mathematical economics, nowadays further developed by his former students and collaborators. His interest was not restricted to scientific research alone, since he was also an excellent expert of china and a co-author of a book on this subject. His wife, dr Maria Wycech-Łoś, co-author of some of their joint publications, firmly supported and accompanied him in his scientific research.

Jerzy Łoś devoted much of his time and effort to the organisation of numerous scientific activities, as well as to the education of young researchers. One of the most relevant achievements in this area was the doctoral course created and run by him in the Institute of Mathematics of the Polish Academy of Sciences from 1967 till 1971. The subject of this course was applications of mathematics in economics and the lecturers were outstanding mathematicians interested in economics and independent economists not necessarily in line with the official dogmas, such as Professors Włodzimierz Brus and Michał Kalecki. The list of students who completed the course with a PhD in mathematics includes: Grażyna Białynicka-Birula (Toruńczyk), Tadeusz Bromek, Stanisław Byłka, Jan Komar, Jolanta Kubala, Longina Mardoń, Andrzej Nowak and Ryszarda Rempała. In the forthcoming years Łoś was still educating and promoting doctors,

whose theses dealt with mathematical economics or mathematics created for the current needs of economics. Their names are: Stanisław Ambroszkiewicz, Lidia Filus, Maria Godlewska (Kędzierska), Adam Idzik, Mirosław Michalik, Adam Piołunowicz, Honorata Sosnowska, Andrzej Wieczorek, Katarzyna Wiśniowolska (Iwaniec) and three students from Vietnam. Łoś also promoted many Masters of Science in mathematics.

One of the most important scientific events organised by Łoś (jointly with an outstanding economist, Professor Cezary Józefiak) was an interdisciplinary research and expository seminar, which gathered economists interested in mathematical methods and mathematicians interested in applications. The activities of the seminar continued for many years in the sixties and seventies. The seminar, of a somewhat dissident character (criticising the regime was allowed and routine), was a rather exceptional event in those years. The list of participants included Professors and Doctors Tadeusz Kowalik, Włodzimierz Hagemajer, Wiktor Herer, Jan Mujżel, Jerzy Osiatyński, Wiesław Sadowski, Zdzisław Sadowski, Paweł Sulmicki, Włodzimierz Trzeciakowski. The seminar was usually held in the Mirror Hall of the Staszic Palace, the main site of the Polish Academy of Sciences.

The International Symposium on Mathematical Methods in Economics, organised by Jerzy Łoś in 1972 at the Institute of Mathematics of the Polish Academy of Sciences was an international event of great importance and influence for the future research in Poland. The symposium, extended over a period of six months, preceded the symposia of the International Banach Center, organized at that time as a part of the activities of the Institute of Mathematics. It was followed by an International Conference on von Neumann Models. The list of over 100 participants from all continents includes Martin Beckmann, Evgenij Dynkin, Tjalling Koopmans, Terje Hansen, Hofng Tuy, Victor Klee, Lionel MacKenzie, Milan Mareš, Otto Moeschlin, Oskar Morgenstern, Steve Robinson, Tyrell Rockafellar and Gerald Thompson.

Another huge conference organised by Jerzy Łoś was devoted to equilibria and finding methods; it took place in Toruń (Poland) in 1974. The two events, the symposium and conference in 1972 and the conference in 1974 have been commemorated by two volumes of research papers, published jointly by North-Holland and the Polish Scientific Publishers PWN and edited by Jerzy Łoś and Maria Wycech-Łoś.

In the seventies Łoś also organised several smaller conferences named „Polish-Soviet Symposia on Mathematical Economics” whose participants were, however, people from other countries as well. In 1976 Jerzy Łoś, jointly with Maria Wycech-Łoś and Andrzej Wieczorek, edited a volume of research papers in mathematical economics which appeared in the Springer series *Lecture Notes in Economics and Mathematical Systems*.

Łoś was always ready and eager to co-operate with researchers and scholars from various countries; he used to convince his young collaborators to promote such co-operation. During his long research visits to the universities in Berkeley (1962/3) and Aarhus in Denmark (1967), to the Poincaré Institute in Paris (1969), Yale (1973) and the University of Wisconsin at Madison (1978/9) he both made research and lectured on mathematical economics and applications of mathematics. He had many friends all over the world, also in the former Soviet Union (many Russian scholars attending some conferences held in Warsaw in those hard times occasionally stayed at his apartment). Jerzy Łoś found particularly close friends in two people: Professor Otto Moeschlin from Hagen in Germany and Professor Steve Robinson from Madison, Wisconsin, who frequently paid him visits in Warsaw.

### **Publications**

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