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Multilateral Ranking Negotiations

Antoni Mazurkiewicz *
amaz@ipipan.waw.pl
Institute of Computer Science of PAS, Warsaw

Abstract. In general, negotiations within a group of participants are processes, that starting with participants in some arbitrary (initial) states eventually will achieve an agreement with all participants being in the required negotiated states. Process of negotiation is performed according to a negotiation protocol. Here, an ordering of participants is taken as the negotiation goal; constructing a negotiation protocol for this purpose is referred to as the ranking problem. The formal method used for discussing the considered issue are local computations; in general, they consist in transforming states of the whole structure by way of transforming states of its substructures. The paper aims to discuss communication structures that admit negotiations limited to direct communications between participants of a single 'association' at a time. Necessary and sufficient conditions for existence of such a ranking protocol for considered structures are formulated and a universal protocol for ranking is given. The paper is a generalization of bilateral negotiations presented in [14], where negotiations are limited to associations containing at most two members; the multilateral protocol presented in this paper covers the case of bilateral negotiations.

1. Introduction.

In [14] (and independently in [2]) local computations on edges of graphs are discussed. Such computations can be viewed as local transformations of graph labels limited to pairs of neighbour nodes. Such local transformations can be compared to negotiations performed by means of classical phone connection: one step of such negotiations leads to state transformation of two partners communicating at the moment. The natural question arises about power (and limitations) of such a strong locality of computations; the papers quoted above give an answer to this question. In the present paper phone connection joining two partners is generalized to meeting of a number of the negotiating participants (in particular,

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Address for correspondence: Institute of Computer Science of PAS, Ordona 21, 01-237 Warsaw

two of them, as in the previous case). It turns out that the idea of the earlier proposed protocols remains almost the same - the generalization makes the formalism even simpler than that used in the former case.

To set up the problem, suppose there is a number of individuals, each of them brings an integer as a label. They are grouped into associations; within an association they can communicate, exchange information, and modify their labels; there is no possibility of direct communication between individuals that do not belong to the same association. However, since some individuals can be affiliated to more than one association, indirect communication between remote individuals is possible using individuals with multiple affiliations as go between. Such systems, built from individuals and their associations will be called here *communication structures*. Associations act by their assemblies that take place from time to time; an association is active during its assembly, and passive out of it. The purpose of an assembly is to exchange information among participants of the active association and update their labelling. The way of information exchange is subjected to some rules collected in the communication protocol.

As a purpose of negotiations we chose a ranking of individuals, i.e. a situation where different individuals are labelled with different integers. Therefore, the general task of the negotiation protocol is to guarantee reaching such a situation. The question arises for which class of communication structures such a protocol exists. This paper addresses this issue; necessary and sufficient conditions for communication structures that make possible construction of the required protocol will be given.

The paper is organized as follows. First, the notion of communication structure is introduced; next, the notion of multilateral protocol is defined and the behaviour of the system according to such protocols is described. Finally, a class of communication structures for which ranking protocols exist is defined. It is proved that for communication structures beyond this class no ranking protocol exists and that for any communication structure within this class such a protocol can be constructed.

In the paper the standard mathematical notation is used. The set of non-negative integers is denoted by $\mathcal{N}$. The set of all mappings from X to Y is denoted by $X \longrightarrow Y$. The terms 'function' and 'mapping' will be used interchangeably. If ρ is a relation, write $x\rho y$ rather than $(x,y) \in \rho$, if more convenient. Sign $\circ$ denotes the composition of relations. A symmetric, transitive, and reflexive relation is an *equivalence*. Any equivalence relation $\simeq \subseteq X^2$ defines a partition of X into its equivalence classes; the class containing x will be denoted by $[x]_{\simeq}$, or $[x]$, if $\simeq$ is understood. By definition of equivalence classes $x \simeq y \Leftrightarrow [x]_{\simeq} = [y]_{\simeq}$. Equivalence classes of an equivalence in set X are non-empty, pairwise disjoint, and covering X.

2. Communication structures.

A family E of sets is said to be *connected*, if for any $e', e'' \in E$ there exists a sequence $(e_0, e_1, \ldots, e_n)$ of members of E such that $e' = e_0, e'' = e_n$, and $e_{i-1} \cap e_i \neq \emptyset$ for $i = 1, 2, \ldots, n$. Observe that all sets in a non-empty connected family are not empty. Let V be a set and E be a covering of V, i.e a family of subsets of V such that $\bigcup E = V$. Any pair $F = (V, E)$, where V is a finite, non-empty set and $E \subseteq P(V)$ is a connected covering of V, will be called here a *communication structure*. Elements of V are called *individuals* and those of E *associations* of F; individuals and associations of a communication structure are its *objects*. An individual belonging to an association is its member. Since E is a covering of V, any individual is a member of at least one association. The number of individuals of a communication structure is its *size* and that of associations is its *degree*. Individuals not belonging to association e are said to be *separated* from e. From the mathematical point of view communication structures can be viewed as finite connected hypergraphs.

Example 2.1. In Fig.1 communication structure with seven individuals $v_1, \ldots, v_7$ and four associations $\{v_1, v_2\}$, $\{v_2, v_3, v_4, v_5\}$, $\{v_4, v_5, v_6\}$, $\{v_4, v_7\}$ is presented in a graphical form.

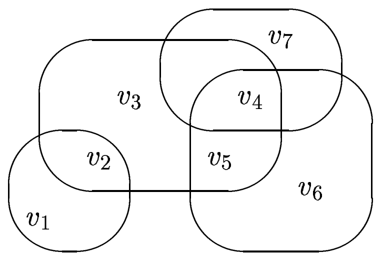

Figure 1. Diagram of a communication structure

Particular types of communication structures are: (i) total communication structures, all individuals of which are members of a single association; (ii) individual communication structures, all associations of which have a common member; (iii) bilateral communication structures, with all associations containing precisely two members (communication structures of this type can be identified with finite undirected graphs). Associations act by their assemblies that take place from time to time; an association is active during its assembly, and passive out of it. The purpose of an assembly is to exchange information among states of participants and update their knowledge about the whole structure state. All association members are obliged to take part in any assembly of this association, but no individual can take part in more than one assembly at the same time. From this it follows that none of associations with a common member can be active simultaneously. The way of information exchange is subjected to some rules collected in the communication protocol.

2.1. Negotiations in communication structures.

The purpose of negotiations is to achieve a consensus, i.e. a common agreement on reached states of negotiation participants. At the beginning of negotiations all participants are in their individual states that, in general, do not fulfil the required conditions. Consensus is a result of series of local exchange of information and local negotiations; the characteristic feature of such negotiations is lack of any arbiter capable to impose the assumed final result to all participants. As we mentioned above, the rule that all negotiating parties are supposed to observe will be called the negotiation protocol. Here, we are interested in protocols with some limitations concerning the number of participants taking part in a single step of negotiations, namely we limit ourselves to discuss negotiations with communications limited to associations of participants; such negotiations are called here *multilateral*. Multilateral negotiation rules can be compared with negotiations made during assemblies of participants; only members of an active association, i.e. association at its assembly, are engaged in the transformation of their individual standpoints into a new common position.

Internal states of participants can contain information of different kinds; in particular, the own state of a participant and its (limited) knowledge about the whole system state. In general, global states of the system are not known to the participants; they know exclusively their own individual states, but those of other participants are known to them only indirectly, by way of a series of communicative acts. The system is acting in a discrete way, changing its states in some discrete moments. A single (atomic) act is either a knowledge exchange between participants, or transformations of their individual states, or common agreement of all association members on changing the name of the active association. In effect of a knowledge exchange the knowledge of both participants is unified and the sequence of such knowledge exchanges is the only way of transmitting information between remote participants. Transformations of individual states are subjected to rules of negotiations that result in a common intermediate agreement between participants of negotiations (and possibly next modified by a successive course of negotiations) The collection of such rules forms a protocol of negotiations. It should be stressed that all transformations are local, i.e. they depend and concern pairs of participants only. Another important feature of negotiation protocols is fairness, i.e independence of rules on particular participants; any transformation of such a protocol can be applied to any participants provided they meet requirements concerning their individual states, but regardless their names. Therefore, any systematic change of participants names does not influence on the protocol course and the choice between a number of candidates to perform an action in a bilateral negotiation is non-deterministic. In the present paper we are interested mainly in logical possibilities and limitations of negotiations subjected to the multilateral communications inside associations. An injective labelling is taken as a paradigm of negotiation target, since once a sequence of participants has been established, other negotiation goals can be easily derived. For instance, knowing ranking positions of participants it is possible to establish for them a ”round robin” access to a system resources, or to elect a leader, or use their sequence for any type of further sequential processing.

2.2. Multilateral ranking protocols.

To put above considerations in a formal way, let $F = (V, E)$ be a communication structure. Individuals in F will represent participants of negotiations, associations in E will represent possibilities of direct connections between participants; two participants can directly communicate only if they are members of the same association. Let S be an arbitrary set; elements of S, called individual labels, can be viewed as individual negotiation positions of participants of negotiations. Any mapping $\sigma : V \cup E \longrightarrow \mathcal{N}$ is a global valuation of the system and can be viewed as an instantaneous state of negotiations, with $\sigma(v)$ representing the individual negotiation position of participant v and $\sigma(e)$ the actual name of association e. The goal of negotiations is to achieve a one-to-one assignment of integers to individuals in V.

Let $F = (V, E)$ be a communication structure, $X = V \cup E$, and S be an arbitrary set (of labels); the global state of the system F is any mapping $\sigma \in X \longrightarrow S$; the set of all global states will be denoted by Σ. Any binary relation $T \subseteq S^* \times S^*$ such that

$$((s'_0, s'_1, s'_2, \ldots, s'_k), (s''_0, s''_1, s''_2, \ldots, s''_m)) \in T \Rightarrow k = m > 0$$

will be called a *transformation rule* for S and the pair (S, T) a *transformation protocol*, or simply a *protocol* for F. Let $P = (S, T)$ be a protocol and $e \in E$ be an association of F. Call elements of $e \cup \{e\}$ objects of e; that is, objects of e are all its elements and e itself. We say that global state σ'' of F arises from another global state σ' of F in effect of assembly of association e and write

$$\sigma' \xrightarrow{e}_P \sigma'', \tag{1}$$

if and only if

$$\exists v_1, v_2, \ldots, v_k \in V : \{v_1, v_2, \ldots, v_k\} = e$$
$$\wedge \ ((\sigma'(e), \sigma'(v_1), \ldots, \sigma'(v_k)), (\sigma''(e)\sigma''(v_1), \ldots, \sigma''(v_k))) \in T \quad (2)$$
$$\wedge \ \forall x \in (V - e) \cup (E - \{e\}) : \sigma''(x) = \sigma'(x). \quad (3)$$

From the above definition it follows that state transformation $\sigma' \stackrel{e}{\rightarrow}_P \sigma''$ is localized to objects of an association of F; states of objects beyond this association are neither influencing on the effect of such transformation (2) nor changed by it (3). The step relation of P for F is now defined as follows:

$$\sigma' \rightarrow_P \sigma'' \Leftrightarrow \sigma' \neq \sigma'' \wedge \exists e \in E : \sigma' \stackrel{e}{\rightarrow}_P \sigma''. \quad (4)$$

Thus, the step relation of P is a state transformation localized to an association. Call a state σ *final*, if there is no state σ' with $\sigma \rightarrow_P \sigma'$; the set of all final states for P will be denoted by $\mathbf{F}$. A run of P is any sequence $r = (\sigma_0, \sigma_1, \ldots, \sigma_i, \ldots)$ of global states such that

$$\forall i \geq 0 : \sigma_i \rightarrow_P \sigma_{i+1} \ \vee \ \sigma_i \in \mathbf{F} \wedge \sigma_i = \sigma_{i+1}. \quad (5)$$

Call run r terminal, if there is $i \geq 0$ such that $\sigma_i \in \mathbf{F}$ and then the run r is said to terminate with σ_i.

Protocol $P = (S, T)$ is an *ranking* protocol for communication structure $F = (V, E)$, if S is a linearly ordered set and the following conditions are satisfied:

Any run of P is terminating; (6)

Any final valuation of P is injective on V. (7)

Then any final valuation r_f of P defines a ranking in V such that v' precedes v'' iff $r_f(v') < r_f(v'')$. In [14] a bilateral universal protocol for graphs has been defined; in the present paper this result is extend to a broader class of structures. This class will be defined in the next section and it will be shown that for none of similar structures beyond this class such a protocol exists.

3. Multilateral similarity.

Let $F = (V, E)$ be a communication structure. Binary relation $\mathbf{A} \subseteq V \times E \cup E \times V$ such that

$$(x, y) \in \mathbf{A} \Leftrightarrow x \in y \vee y \in x \quad (8)$$

is called the *affiliation* relation in F. Clearly, $\mathbf{A}$ is a symmetric and irreflexive binary relation in V. If $(x, y) \in \mathbf{A}$, we say that x is affiliated to y; observe that, not only individuals are affiliated to their association, but also associations are affiliated to their members, opposing to the colloquial language.

Equivalence relation $\sim$ in $V^2 \cup E^2$ is called a *multilateral similarity*, or simply, *similarity*, if it meets the following two conditions:

$$(x, z) \in \mathbf{A} \wedge (y, z) \in \mathbf{A} \wedge x \sim y \Rightarrow \ x = y, \quad (9)$$
$$(x, z) \in \mathbf{A} \wedge z \sim y \ \Rightarrow \ \exists z' : (y, z') \in \mathbf{A} \wedge z' \sim x, \quad (10)$$

or, in a more compact form,

$$\mathbf{A}^2 \cap \sim = \mathbf{I}, \tag{11}$$

$$\mathbf{A} \circ \sim = \sim \circ \mathbf{A}. \tag{12}$$

Any two objects x, y of G with $x \sim y$ are said to be *similar*. Condition (9) means that different individuals in the same association as well as different associations with a common member are not similar, and (10) similar objects have similar affiliated objects. The diagram shown in Figure 1 is an illustration of condition (10).

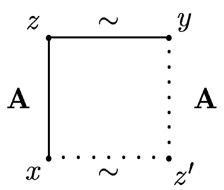

Figure 2. Illustration of condition (10)

Global state σ of X is injective, if

$$\forall (x, y) \in V^2 \cup E^2 : \ \sigma(x) = \sigma(y) \ \Rightarrow \ x = y, \tag{13}$$

and σ is uniform, if

$$\forall (x, y) \in V^2 \cup E^2 : \ \sigma(x) = \sigma(y). \tag{14}$$

Call a labelling $\phi \in V \cup E \longrightarrow S$ of objects of communication structure $F = (V, E)$ *locally injective*, if there is a similarity $\sim$ such that for all objects $(x, y) \in V^2 \cup E^2$

$$x \sim y \Leftrightarrow \phi(x) = \phi(y). \tag{15}$$

It is clear that for any similarity $\sim$ in F such a locally injective labelling exists. It is also clear that any injective labelling is a particular case of a locally injective labelling.

Theorem 3.1. Let F be a communication structure and let $\sim$ be a similarity in F. Let $[x]$ denote the equivalence class of $\sim$ containing x. Then for any objects x, y of F:

$$\mathrm{card}[x] = \mathrm{card}[y]. \tag{16}$$

Proof:
Let $\sim$ be a multilateral similarity in communication structure F. First prove that for all objects x, y of F such that $(x, y) \in \mathbf{A}$ we have $\mathrm{card}[x] = \mathrm{card}[y]$. Indeed, let $(x, y) \in \mathbf{A}$ and $x \sim x'$, $x \neq x'$; then by (10) there is also y' such that $y \sim y'$; moreover, $x' \neq y'$, since otherwise it would be $(x, y) \in \mathbf{A}$, $(y, x') \in \mathbf{A}$ giving $(x, x') \in \mathbf{A}^2 - \mathbf{I}$ and contradicting by (9) to $x \sim x'$ (see Figure 2). From this it follows that the number of objects similar to y is not less than the number of objects similar to y, i.e. $\mathrm{card}[x] \leq \mathrm{card}[y]$. Interchanging x with y (by symmetry of $\mathbf{A}$ relation) we get $\mathrm{card}[y] \leq \mathrm{card}[x]$, hence $\mathrm{card}[y] = \mathrm{card}[x]$. Now, let x, y be arbitrary objects of F; since $(x, y) \in \mathbf{A}^*$ by connectivity of F, $\mathrm{card}[x] = \mathrm{card}[y]$ holds for arbitrary objects of F. □

Since the number $\mathrm{card}[x]$ is the same for all objects of a communication structure and depends solely on the similarity relation, call it the *index* of the similarity. It is clear that the ranking relation is a multilateral similarity with index 1, and that for any symmetric communication structure there exists multilateral similarity with index > 1. Communication structure F is *prime*, if any similarity in F is the ranking relation. As it follows from Example 3.1, it is not so the other way round: there are communication structures admitting similarity relations different than the ranking (or, equivalently, admitting locally injective valuations that are not injective). Such structures will be called *symmetric*. A simple example of a symmetric structure is presented in 3.1.

Example 3.1. Let the communication structure $F = (V, E)$ be defined by

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}, \tag{17}$$

$$E = \{\{v_8, v_1, v_2\}, \{v_2, v_3, v_4\}, \{v_4, v_5, v_6\}, \{v_6, v_7, v_8\}\}. \tag{18}$$

The least equivalence $\sim$ containing the pairs (v_i, v_{i+4}) $(i = 1, 2, 3, 4)$, (e_1, e_3), (e_2, v_4) is a bilateral similarity. Let valuation φ be defined by $\varphi(v_i) = \varphi(v_{i+4}) = i$ $(i = 1, 2, 3, 4)$, $\varphi(e_1) = \varphi(e_3) = a$, $\varphi(e_2) = \varphi(e_4) = b$ and equivalence $\sim$ meets the condition $x \sim y \Leftrightarrow \varphi(x) = \varphi(y)$ (see Fig.2).

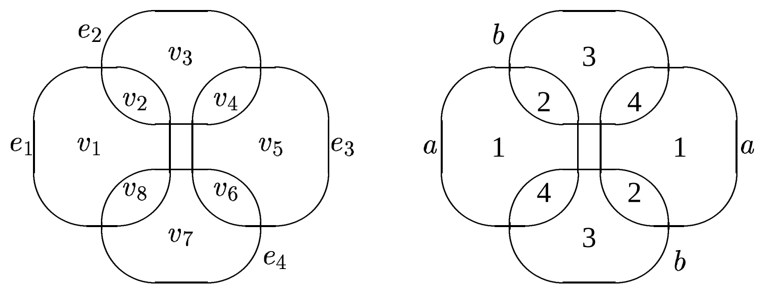

Figure 3. A symmetric communication structure with a locally injective labelling

Labelling φ is locally injective, but not injective, proving F to be symmetric.

Proposition 3.1. Communication structure $F = (V, E)$ is symmetric if and only if there exists a locally injective valuation φ of F such that $\mathrm{card}(\varphi(V)) < \mathrm{card}(V)$ and $\mathrm{card}(\varphi(E)) < \mathrm{card}(E)$.

Proposition 3.2. Any total communication structure is prime (all individuals of such a structure belong to the same association); any individual communication structureis prime (there is an individual belonging to all associations of such a structure).

Theorem 3.2. For any communication structure $F = (V, E)$ and any similarity relation $\sim$ in F with index k numbers $\mathrm{card}(V)$ and $\mathrm{card}(E)$ are divisible by k.

Proof:
Let k be the index of a similarity relation in communication structure (V, E). Then $\mathrm{card}[x] = k$ for any object of G. By definition of similarity, equivalence class of an individual contains exclusively individuals, and equivalence class of associations contains only associations. From this it follows that $\mathrm{card}(V) = m \cdot k$ and $\mathrm{card}(E) = n \cdot k$ for some integers m, n. It proves the theorem. □

Corollary 3.1. Communication structures with prime size or with prime degree are prime; communication structures with size and degree prime to each other are prime.

Example 3.2. The degree of the communication structure given in Example 2.1 is 7, hence it is prime; it means that the whole communication structure is prime.

Graphs can be viewed as particular cases of communication structures, where each association contains precisely two members. For this type of structures we have the following property, resulting directly from the above theorem.

Corollary 3.2. Graphs with prime size or with prime degree are pairwise prime (cf. [11]); graphs with size and degree prime to each other are pairwise prime; in particular, trees are pairwise prime (the number of nodes of a tree and that of its associations differs by 1, hence they are relatively prime).

Theorem 3.3. There is no ranking protocol for any symmetric communication structure.

Proof:
Let $F = (V, E)$ be a communication structure, $X = V \cup E$, and let $\sim$ be a similarity relation in F with index $k > 1$, arbitrary but fixed for the rest of the proof. Call a labelling $\phi \in V \longrightarrow S$ symmetric, if $\phi(x) = \phi(y)$ for any pair of individuals $x, y \in V$ such that $x \sim y$. Notice that none of symmetric valuation can be injective, by its very definition. Suppose $P = (S, T)$ is an arbitrary ranking protocol. Construct a sequence of labellings as follows. Let σ_0 be a symmetric labelling. Suppose $i \geq 0$ and $\sigma_i \in X \longrightarrow S$ is a symmetric labelling obtained by a sequence of rules of P; then, since $k > 1$, s_i cannot be injective. It it were final, condition (i) for the ranking protocol would fail, as terminating with a valuation which is not injective. Suppose σ_i is live; then there is association $e^{(1)}$ and rule $r \in T$ such that r can be applied to $e^{(1)}$ transforming it into a new valuation. Let $e^{(1)}, e^{(2)}, \ldots, e^{(k)}$ be all associations of G similar to $e^{(1)}$; since valuation σ_i is symmetric and similar associations have similar members, by the definition of transformation rules, the same rule r can be successively applied to all associations $e^{(1)}, e^{(2)}, \ldots, e^{(k)}$ giving in effect a new symmetric valuation σ_{i+1}. Thus, given a symmetric valuation which is not final, after k steps of the protocol a new symmetric valuation is obtained. Therefore, either a symmetric valuation is final, hence contradicting (7), or, by induction, there exists an infinite run, contradicting to (6). Thus, no ranking protocol for F exists. It completes the proof. □

4. The universal ranking protocol.

In this section a multilateral ranking protocol for communication structures will be defined; it will be shown that for each prime structure this protocol meets conditions (6) and (7) required for such protocols. The idea of such a ranking protocol is quite simple. We assume that in a single step of the protocol only one association is subjected to processing. Call such an association active and its members active individuals in such a step. We start with an arbitrary assignment of numbers to individuals and to associations of the considered system. Unless the assignment is a required enumeration, there exists an active association. If two active individuals are labelled with the same number, the label of one of them is increased; if the active association is labelled with the same number as an associations to which an active individual belongs, the label of the active association is increased. If an active individual ”knows” about a remote individual with the same label, its label can be increased, provided the remote individual has a ”stronger” neighbourhood than the active one; if an active individual ”knows” about another remote association with the same label, the label of the active association can be increased, provided the remote association has ”stronger” members than the active one. The strength of objects will be defined in the sequel in such a way that it increases together with increasing object labels. This construction guarantees that changing the label of one object remains another object with the previous label, consequently protecting labels of remote objects against unbounded growth. In this way the main problem in local processing, namely communication between remote objects, is avoided. Below a more precise definition of the protocol is described.

First, some additional tools and facts are given. One of them is total ordering of finite ”collections” of integers. Collections of objects are sets of occurrences of objects, not of objects themselves; as opposed to sets, collections can contain a number of occurrences of the same object.

4.1. Collections of integers and their ordering.

Let Z be a set. By a *collection* of elements of Z we shall mean here any mapping of a finite set (of indices) into Z. Collection $\phi \in I \longrightarrow Z$ such that $\phi(i) = z_i$ will be denoted by $\langle z_i \mid i \in I\rangle$; collection $\langle z_i \mid i \in \{1, 2, \dots, n\}\rangle$ will be denoted by $\langle z_1, z_2, \dots, z_n\rangle$. Two collections $\phi' \in I \longrightarrow Z, \phi'' \in J \longrightarrow Z$ are isomorphic (and identified), if there is a bijection $\psi \in I \longrightarrow J$ such that $\phi'(i) = \phi''(\psi(i))$ for all $i \in I$. The set of all (isomorphism classes of) collections of elements of Z is denoted by $\mathbf{C}(Z)$. Since the set of indices is irrelevant in the collection definition, it will be omitted whenever it is possible.

Example 4.1.

$$A = \langle 1, 3, 2, 4\rangle, \qquad B = \langle 1, 3, 2, 3, 4\rangle$$

are two different collections of integers.

The empty collection, i.e. a collection with the empty set of indices, is denoted by $\langle\rangle$. An element x is in collection $A = \langle a_i \mid i \in I\rangle$, in symbols $x \in A$, if there is $j \in I$ such that $a_j = x$. Let $A = \langle a_i \mid i \in I\rangle$ be a collection of elements of Z and $x \in Z$. The result $A - x$ of deleting x from A is either A, if $x \notin A$, or $A' = \langle a_i \mid i \in I - \{i_0\}\rangle$, if $a_{i_0} = x$ (notice that in this case only one occurrence of x is deleted). For example, $\langle 1, 2, 3, 2, 4\rangle - 2 = \langle 1, 3, 2, 4\rangle$. Difference $A - B$ of two collections A and B is defined recursively:

$$A - \langle\rangle = A, \tag{19}$$

$$A - \langle b_0, b_1, \dots, b_n\rangle = (A - b_0) - \langle b_1, b_2, \dots, b_n\rangle, \tag{20}$$

e.g $\langle 1, 2, 3, 2, 4\rangle - \langle 2, 3, 5\rangle = \langle 1, 2, 4\rangle$. In what follows only collections of integers will be considered. Let binary relation $\prec$ in the set of all collections of non-negative integers be defined by the equivalence:

$$A \prec B \;\Leftrightarrow\; \exists x \in B - A : \; \forall y \in A - B : \; x < y, \tag{21}$$

e.g. $\langle 1,2\rangle \prec \langle 3\rangle$, $\langle 1\rangle \prec \langle 1,1\rangle$, and $\langle 1,2,3,2,4\rangle \prec \langle 2,3,5\rangle$. If $A \prec B$, then collection B is said to be stronger than A.

Proposition 4.1. Relation $\prec$ is a total order in the set $\mathbf{C}(\mathcal{N})$.

Proof:
To prove transitivity of $\prec$ assume $A \prec B \prec C$ and let $a \in A - C$. It suffices to prove that there is $c \in C$ with $c > a$. There can be two cases: (1) either $a \in A - B$, or (2) $a \in B$. In case (1) there is $b \in B$ with $b > a$ and there is $c \in C$ with $c > b$, hence $c > a$. In case (2) $a \in B - C$, hence there is $c \in C - B$ with $c > a$. It proves transitivity. To prove irreflexivity, suppose $A \prec A$; then there would be $(\exists n \in \langle\rangle)(\forall m \in \langle\rangle) n > m$, which is evidently false. Let now $A \neq B$, i.e. either $A - B \neq \langle\rangle$, or $B - A \neq \langle\rangle$. If $A - B = \langle\rangle$, then each element in $B - A$ is greater that any element of $A - B$ (which is empty) which proves $A \prec B$. Similarly, if $B - A = \langle\rangle$, then each element in $A - B$ is greater that any element of $B - A$ (which is empty in turn) which proves $B \prec A$. If neither $A - B = \langle\rangle$, nor $B - A = \langle\rangle$, then there is $a \in A - B$ which is greater than any element of $B - A$ and then $A \prec B$, or there is $b \in B - A$ which is greater than any element of $A - B$ and then $B \prec A$. Therefore in any case either $A \prec B$ or $B \prec A$. It ends the proof. □

Set
$$A \preceq B \Leftrightarrow A \prec B \vee A = B.$$
Clearly, $\preceq$ is a weak linear order in $\mathbf{C}(\mathcal{N})$. Observe that $\emptyset \prec A$ for all non-empty collections A of integers and that for any set of non-negative integers $\{b_i \mid i \in I\}$ such that $a_i \leq b_i$ for each $i \in I$ the relation
$$\langle a_i \mid i \in I\rangle \preceq \langle b_i \mid i \in I\rangle \tag{22}$$
holds. This property will be used in the sequel. It says that replacing in collection A elements with greater elements results in a collection not weaker than A.

Let Z', Z'' be two collections of non-negative integers. Let $\langle z'_1, z'_2, \ldots, z'_k\rangle$, $\langle z''_1, z''_2, \ldots, z''_m\rangle$ be representations of Z', Z'', respectively, such that $z'_{i-1} \geq z'_i$, $z''_{i-1} \geq z''_i$ for all $i > 1$. It is worthwhile to note then $Z' \prec Z''$ if and only if the sequence $z'_1, z'_2, \ldots, z'_k$ precedes lexicographically $z''_1, z''_2, \ldots, z''_k$. In fact, lexicographic ordering of sequences could be taken as well as a basis of collections ordering.

4.2. Structure of the universal protocol.

Universal multilateral ranking protocol is a scheme intended to equip any concrete communication structure with a (concrete) multilateral protocol. Before defining it some auxiliary definitions are needed. Let $F = (V, E)$ be an arbitrary communication structure fixed for this section, $X = V \cup E$, and let S be defined as
$$S = \mathcal{N} \times (X \longrightarrow \mathcal{N}).$$
Consider set $\Sigma = X \longrightarrow S$ of valuations of F. Let $\sigma \in \Sigma$ with $\sigma(x) = (N(x), K(x))$, where $N(x) \in \mathcal{N}$ and $K(x) \in X \longrightarrow \mathcal{N}$; call $N(x)$ the (current) *name* of object x at σ and the mapping $K(x)$ the (current) *knowledge* of x about names of all objects of F at valuation σ. Write $K(x, y)$ for $K(x)(y)$. If $K(x)$ is the knowledge of x, then $K(x, y)$ is the label of y known to x. Define the ordering in $X \longrightarrow \mathcal{N}$ by the equivalence:
$$N' \leq N'' \Leftrightarrow \forall x \in X : N'(x) \leq N''(x),$$

and similarly the ordering of valuations:

$$(N', K') \leq (N'', K'') \Leftrightarrow \forall x, y \in X : N'(x) \leq N''(x) \wedge K'(x, y) \leq K''(x, y).$$

To avoid confusion, mappings in $X \longrightarrow \Sigma$ will be called valuations, while those in $X \longrightarrow \mathcal{N}$ labellings. Thus, any valuation of an object is a pair (current name, current knowledge), and any labelling is an assignment of names to the structure objects. The current knowledge of an object about names of other objects is then a labelling. For a given labelling $K \in X \longrightarrow \mathcal{N}$, we say that object y is *stronger* than object x, if the relation $x \prec_K y$ defined by

$$x \prec_K y \;\;\Leftrightarrow\;\; \langle K(z) \mid (x, z) \in \mathbf{A} \rangle \prec \langle K(z) \mid (y, z) \in \mathbf{A} \rangle$$

holds.

Example 4.2. In Figure 4 the strength of two individuals and in Figure 3 the strength of two associations are compared, assuming the labelling of corresponding objects is as shown in the picture.

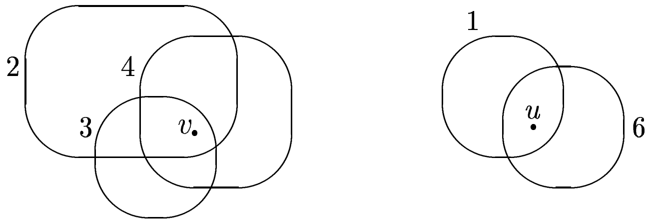

Figure 4. Strength of individuals: $v \prec u$ (u is stronger than v)

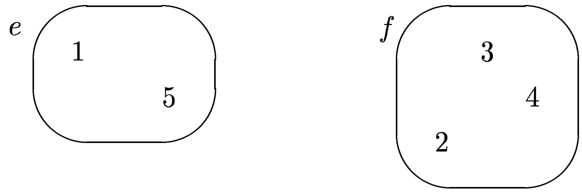

Figure 5. Strength of associations: $f \prec e$ (e is stronger than f)

4.3. Universal protocol rules.

Universal protocol R is the pair (S, T) with S defined as above and T (the set of rules of R) defined as follows. Let e be an arbitrary association of F; then application of a rule of T to association e, called below the *active* association, consists in:

1. (**Knowledge unification**.) The knowledge of all members of e (called then *active* individuals in the sequel) become unified by taking the maximum of their knowledge; this unified knowledge will be called the common knowledge of e. In all subsequent steps but the last one all updating will refer to and concern this common knowledge only; these updating steps can be performed or skipped.

2. (**Association name updating**.) Check the names of associations of F in the common knowledge of individuals in e; if, according to this knowledge there exists a different association with the same name as the active one (called below its ”twin association”), proceed to the next step (or skip it).

3. Increase the name of the active association in one of the following cases:

   a. (Direct) There exists an active individual belonging to the twin association;

   b. (Remote) Its twin association is disjoint with e, but stronger than e.

4. (**Individual name updating**.) Check the names of individuals of F (known to the active individuals); if, according to the common knowledge of active individuals, there exists another individual with the same name as the active one (called below its ”twin individual”), proceed to the next step (or skip it).

5. Increase the name of the active individual in one of the following cases:

   a. (Direct) Its twin individual is in e (is active);

   b. (Remote) Its twin individual is separated from e, but stronger than the active one.

6. (**Common knowledge updating**.) Update the common knowledge (of e) by the newly introduced names, assign the updated knowledge to all active individuals, then close the procedure.

After completing this procedure association e is de-activated (becomes passive). The following observations concerning the above protocol rules can be made. Firstly, all updating concern valuations of active objects only. Secondly, knowledge about other objects of the processed structure is gained via transmission of local knowledge only. This knowledge is spread out by its unifications made by the first step of the rule. Thirdly, all names of objects occurring in valuations can only increase; in particular, knowledge unification performed by taking its maximum makes the knowledge increasing. The last observation: all above updating can be performed or skipped. Notice also that a label of an object can be changed only if there exists another object with the same label and remaining unchanged.

It will be shown in the sequel that (i) only a finite number of updating can be performed, which proves the termination of the protocol activity, and (ii) the terminal labelling is either bijective, hence defining a ranking, or it establishes a symmetry of the communication system, hence proves that in this case any other protocol can fail as well.

To describe rules of the protocol in a more formal way, formulate them in the following description, illustrated in Table 1.

Types of variables N, K, and M are the same as defined in the above informal description; the structure of the procedure reflects also the above description. No special initial conditions of the protocol data are assumed. Hopefully, this description is self-explanatory, only instructions of type **for** $x \in X$ and of type $\mathbf{inc}(x)$ need some comments. Namely, the clause following **for** $x \in X$ should be executed 0 or more times, every time for a different element x of X. The choice of elements of X for which the clause has to be executed is non-deterministic. Instruction $\mathbf{inc}(x)$ refers to integers and causes increasing the current value of x by an arbitrary positive value (creating in this way an additional source of non-determinism).

type $v, u \in V;\ e, f \in E; X = V \cup E$
    $K \in X \longrightarrow (X \longrightarrow \mathcal{N});$
    $N \in X \longrightarrow \mathcal{N};$
for $e \in E$
    begin
    type $C \in X \longrightarrow \mathcal{N};$
1: $C \leftarrow \max\{K(x) \mid x \in e \cup \{e\}\}$
    for $f \in E - \{e\}$
2:         **if** $C(e) = C(f) \wedge$
3a: $(e \cap f \neq \emptyset \vee$
3b: $e \cap f = \emptyset \ \wedge\ e \prec_C f)$
            $\mathbf{then\ \ inc}(N(e))$
        end $\{f\}$
    for $v \in e$
        for $u \in V - \{v\}$
4:             **if** $C(v) = C(u) \wedge$
5a: $(u \in e \vee$
5b: $u \notin e \ \wedge v \prec_C u)$
                $\mathbf{then\ \ inc}(N(v))$
            $C(v) \leftarrow N(v)$
            end $\{u\}$
6:         $K(v, v) \leftarrow N(v)$
        end $\{v\}$
7: $K(e, e) \leftarrow N(e)$
    $C(e) \leftarrow N(e)$
    end $\{e\}$

Table 1. Description of the protocol rule for R

5. Properties of the protocol.

Let the denotations used below be the same as in the previous section.

Lemma 5.1. $\sigma' \to \sigma'' \;\Rightarrow\; \sigma' < \sigma''$.

Proof:
By inspecting the protocol rules it is easy to see that no protocol action can decrease any label of an object of the communication structure neither in the structure nor in the knowledge of objects. Since $\sigma' \to \sigma'' \;\Rightarrow\; \sigma' \neq \sigma''$, the implication is proved. □

Let $r = (\sigma_0, \sigma_1 \ldots, \sigma_n, \ldots)$ be a run of protocol R, $\sigma_i = (N_i, K_i)$. Define $\kappa_i \in V \cup E \longrightarrow \mathcal{N}$ by the equality

$$\kappa_i(x) = \max\{K_i(v, x) \mid v \in V\}, \quad \text{for all } x \in V \cup E.$$

In other words, $\kappa_i(x)$ is the greatest label assigned to x known to individuals in V at valuation σ_i. From Lemma 5.1 it follows immediately that for run $(\sigma_0, \sigma_1 \ldots, \sigma_n, \ldots)$, all objects $x \in X$, and all $i \geq 0$

$$\kappa_i(x) \leq \kappa_{i+1}(x).$$

Lemma 5.2. Let $X = V \cup E$. For all $i \geq 0$: $(\forall x \in X)(\exists y \in X)\ \kappa_i(x) = \kappa_{i+1}(y)$.

Proof:
Let $x \in V \cup E$, $\kappa_i(x) = n$. Then there is an object y such that $K_i(y, x) = n$. Let $C_i(x) = \max\{K_i(y, x) \mid y \in X\}$. By definition of κ we have $C_i(x) = n$. Let x_0 be such that $C_i(x_0) = n$ and $C_i(x) = n \;\Rightarrow\; x \prec_{C_i} x_0$ (x_0 is the strongest object x with $C_i(x) = n$). Then no rule of the protocol can rename x_0; it means that $N_{i+1}(x_0) = N_i(x_0)$ and $K_{i+1}(x_0, x_0) = N_i(x_0) = n$; moreover, there is no x with $K_{i+1}(x, x_0) > K_{i+1}(x_0, x_0)$. Thus, $\kappa_{i+1}(x_0) = n$ which proves the lemma. □

From this it follows, loosely speaking, that once a label has appeared as a value of κ_i, it will occur forever (i.e. for all $j > i$), may be as a label of another object.

Lemma 5.3. Any run of protocol R terminates.

Proof:
Let $\kappa_i(x)$ be defined for all $x \in V \cup E$ as above. By By Lemma 5.1 $\kappa_i(x) \leq \kappa_j(x)$ for all $i \leq j$. On the other hand, by Lemma 5.2, κ_i admit only a finite number of different values (namely, not more than $\mathrm{card}(V)+\mathrm{card}(E)$). From this it follows that there exists $j \geq 0$ such that $\kappa_j = \kappa_{j+k}$ for all $k \geq 0$. Since all labels occurring in the protocol valuations are bounded by κ_j, and all of them can only increase, there is also a finite number of them. Thus, there exists $n \geq 0$ such that $\sigma_n = \sigma_{n+k}$ for all $k \geq 0$. It means that σ_n is final; otherwise it would exist σ_{n+1} with $\sigma_n \to \sigma_{n+1}$, hence $\sigma_n < \sigma_{n+1}$, a contradiction. □

The assertion of the following Lemma guarantees equality of knowledge of all system objects as well as its connection with object names.

Lemma 5.4. If $\sigma_f = (N_f, K_f)$ is a final valuation of R for communication structure $F = (V, E)$, then $K_f(x') = K_f(x'')$ for all $x', x'' \in V \cup E$; moreover, $N_f(x) = K_f(x, x)$ for each $x \in E \cup E$.

Proof:
Suppose $K_f(v') \neq K_f(v'')$ for some $v', v'' \in V$. By definition of communication structure E is a connected family, hence there would be $u', u'' \in V$ and $e \in E$ such that $u', u'' \in e$ and $K_f(u') \neq K_f(u'')$. It implies existence $x \in V \cup E$ such that $K_f(u'', x) \neq K_f(u', x)$. Without loss of generality assume that $K_f(u'', x) > K_f(u', x)$. Then by knowledge unification in association e (clause 1 of the protocol rule) K_f could not be the final knowledge in F and then (N, K_f) would not be a final valuation, a contradiction. Equality $N_f(v) = K_f(v, v)$ follows directly from the renaming action (clause 6 of the protocol rule). □

Let (N_f, K_f) be a final valuation of R for communication structure (V, E). The following theorem is the main result of the paper.

Theorem 5.1. Relation $\sim \subseteq V^2 \cup E^2$ defined by equivalence:

$$x \sim y \;\Leftrightarrow\; N_f(x) = N_f(y)$$

for all objects x, y is a similarity relation in F.

Proof:
Observe first that $(x, y) \in \mathbf{A}^2 \wedge x \neq y \;\Rightarrow\; N_f(x) \neq N_f(y)$; otherwise, by clauses 3a or 5a, valuation $\sigma_f = (N_f, K_f)$ of (V, E) could not be final. Assume $N_f(x') = N_f(x'')$ and $(x', y') \in \mathbf{A}$. Then there must be y'' such that $(x'', y'') \in \mathbf{A}$ and $N_f(y') = N_f(y'')$; otherwise, it would be $\langle N_f(y) \mid (y, x') \in \mathbf{A}\rangle \neq \langle N_f(y) \mid (y, x'') \in \mathbf{A}\rangle$. By properties of $\prec$ relation, either $\langle N_f(y) \mid (y, x') \in \mathbf{A}\rangle \prec \langle N_f(y) \mid (y, x'') \in \mathbf{A}\rangle$, or $\langle N_f(y) \mid (y, x'') \in \mathbf{A}\rangle \prec \langle N_f(y) \mid (y, x') \in \mathbf{A}\rangle$. In either case, by clauses 3b or 5b, valuation $\sigma_f = (N_f, K_f)$ could not be final (either label $N_f(x')$ or label $N_f(x'')$ should be changed). Therefore, both conditions for similarity, namely (9) and (10), hold and consequently $N_f(x)$ defines similarity in (V, E). It ends the proof. □

The following result has been obtained with more restrictive requirements (assuming simultaneous actions of all associations containing an individual) and published in [12] and [13]; substantial improvements and converting the proposed protocol into a self-stabilizing one is due to Godard in [6] and [7].

Lemma 5.5. Protocol R is a ranking protocol for any prime communication structure F.

Proof:
Since $F = (V, E)$ is pairwise prime, any similarity in F is the identity relation in $V^2 \cup E^2$. If $\sigma_f = (N_f, K_f)$ is the final valuation of a run of R, then by Theorem 5.1 N_f restricted to V is injective. It means that relation $<$ in V defined by

$$v' < v'' \Leftrightarrow N_f(v') < N_f(v'')$$

is a ranking. □

Observe that in case of symmetric communication structure, the final valuation is either a ranking, or gives a proof of symmetry of the processed communication structure (via mapping defining a similarity relation).

Theorem 3.3 together with Lemma 5.5 give complete characterization of ranking possibilities that can be performed by multilateral negotiations.

Theorem 5.2. There exists a multilateral ranking protocol for communication structure F if and only if F is a prime communication structure.

We close this paper with remarks on fairness of the universal negotiation protocol. Call a mapping uniform, if it has a common value for all arguments. Let $F = (V, E)$ be a communication structure, $X = V \cup E$, and $\sigma \in X \longrightarrow \mathcal{N}$ be a labelling; σ is uniform, if $\sigma(x') = \sigma(x'')$ for any objects x', x'' of F.

Protocol $P = (S, T)$ is *fair*, if for any linear ordering $<$ of V and any uniform valuation $r_p \in V \longrightarrow \{s_0\}$ there exists a run of P starting from r_p and terminating with r_f with $r_f(v') < r_f(v'')$ iff $v' < v''$. Observe that the assumed uniformity of starting valuation is relevant. Without this assumption, one could expect a run of the protocol starting from one one-to-one valuation and terminate with another. However, such an expectation contradicts the properties of runs, since it would enable infinite runs of the protocol, against the protocol requirements.

Theorem 5.3. Universal protocol R is fair.

Proof:
Let $F = (V, E)$ be a communication structure, $V = \{v_1, v_2, \ldots, v_m\}$, and let $v_1 < v_2 < \cdots < v_m$ be the required ranking. Without loss of generality we can assume $r_p(v) = 1$ for all $v \in V$. We prove that there is a run of the universal protocol terminating with r_f such that $r_f(v_i) = i$, hence defining the required ranking. First prove that for any finite and connected family E of sets and any $e_0 \in E$ there exists sequence $(e_1, e_2, \ldots, e_K)$ such that $\{e_1, e_2, \ldots, e_K\} = E$ and

$$\forall\, 1 \le i < K : \; \exists\, j > i : \; e_i \cap e_j \neq \emptyset \wedge \; e_K = e_0. \tag{23}$$

If $K = 1$, then (23) trivially holds. Suppose (23) holds for $K = k > 1$; we prove it for $K = k + 1$. Let then $\mathrm{card}(E) = k + 1$. Since E is connected and $\mathrm{card}(E) > 1$, there are at least two sets $e', e'' \in E$ such that $E - \{e'\}$ as well as $E - \{e''\}$ is connected; as e' chose that one which is not equal to e_0. Let now $(e_1, e_2, \ldots, e_{K-1})$ be the required sequence for $E - \{e'\}$; then $(e', e_1, e_2, \ldots, e_{K-1})$ is the required sequence for E. By induction, our claim holds for any K, i.e. for any finite family.

Now, let the sequence $(e_1, e_2, \ldots, e_K)$ with $\{e_1, e_2, \ldots, e_K\} = E$ and $v_1 \in E_K$ meets condition (23). Describe the run of the universal protocol starting with r_p and terminating with r_f. Let sets $E_1, E_2, \ldots, E_K$ of individuals be defined as

$$E_i = e_i - \bigcup_{j=i+1}^{K} e_j \;\; \text{for } i = 1, 2, \ldots, K;$$

then activate association e_1 and increase (by rule 5a) name $N(v_j) = 0$ of any individual $v_j \in E_1$ to $N(v_j) = j$. It is possible, since e_1 contains "twin" individuals in $V - E_1$ still named with 0. In general, if all individuals in $E_1, \ldots, E_{i-1}$ have been properly renamed, activate association e_i and using rule 5a assign integers j to all individuals v_j in E_i named with 0 as yet. Finally, activate e_K and increase names of all individuals v_j in it but v_1, which still remains with value 1. After doing it, continue the protocol action to its end; however, since all individuals have different names, they are not subjected to be renamed any more. In such a way all individuals are ordered; here, final names of associations are of no importance for the ranking of individuals. □

Notice that the construction of a sequence satisfying (23) made possible to limit all renaming actions of the chosen run to the case described by clauses 3a, without any reference to remote individuals and associations. In fact, it was a purpose of introducing such sequences. Observe also that the above theorem guarantees ranking for an arbitrary communication structure, symmetric as well as prime; reaching desired ranking is always possible by the construction given above.

6. Conclusions.

A criterion for a possibility of ordering individuals of communication structure by local computations with atomic actions limited to associations of individuals (such local computations are called here multilateral negotiations) has been formulated. It is proved that a total ordering of individuals can be achieved by such computations if and only if considered communication structures are prime, i.e. if and only if any equivalence relation in the set of a communication structure objects (individuals and associations) respecting membership relation between objects and not identifying neither objects in the same association nor associations with a common member is the identity relation. This result is a generalization of some earlier results discussing atomic actions concerning graphs and neighborhoods of nodes in graphs.

It is worthwhile to note that the common knowledge of individuals about the system as a whole is here a combination of common knowledge about the basic properties of the protocol (growing property of labels and of number of different labels used in the system labelling). Therefore, if all individual find the number of different labels used in the system labelling equal to the cardinality of the set of objects (associations and individuals), all of them know the protocol has been successfully terminated. If, however, the number of used labels is less than this cardinality, they do not know whether the system is symmetric and no further action is possible, or other associations are still active and a successful termination can be expected in some future.

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