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MONOGRAPH SERIES: FUZZY NUMBERS APPROXIMATIONS, RANKING AND APPLICATIONS

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Introduction

In 1965 Lotfi A. Zadeh published his seminal paper "Fuzzy sets" [202] which opened a new epoch in many disciplines including logic, mathematics, computer science and technology. Suggesting a simple yet convincing formal model to capture imprecision he delivered not only a brilliant idea for theoretical investigations but also effective tools for many practical applications in all those areas where we deal with a natural language and perceptions. The rapid and versatile development of the artificial intelligence and its applications would be hardly imaginable without fuzzy logic and fuzzy systems. Nowadays, in her 50th anniversary, fuzzy set theory still remains one of the most important and promising approach for processing and management of uncertain, incomplete, imprecise or vague information.

Fuzzy number is a fuzzy quantity representing a generalization of a real number. Hence fuzzy numbers might be perceived as one of the most important concept in fuzzy set theory, similarly as ordinary numbers are fundamental objects in classical mathematics. This generalization, although conceptually simple and natural, brings new problems that do not occur in the world of real numbers. For example, fuzzy numbers, contrary to real numbers, are not linearly ordered. Therefore, ranking or ordering fuzzy numbers turns out to be a true challenge. Another area requiring original research unlike usual numbers is connected with the necessity of fuzzy number approximation in order to simplify their shapes for their further calculations, processing and potential software implementation. Thus theory of fuzzy numbers abounds in many interesting problems which are worth studying. And this is the main reason that has motivated us to write down this monograph.

The material in this book has been chosen to provide both the background of fuzzy number theory and snapshot of the state-of-art. In Chapter 1 we present basic information on fuzzy numbers including their representations and characteristics, main families of fuzzy numbers and operations on fuzzy numbers. In Chapter 2 we briefly describe some generalizations of fuzzy numbers. Chapter 3 is devoted to various approximations of fuzzy numbers. In Chapter 4 we consider ranking fuzzy numbers, while in Chapter 5 we discuss some applications of fuzzy numbers. The exercises at the end of each chapter may help the reader to deepen his understanding of the topic.

We would like to express our thanks to Professor Lotfi A. Zadeh for his constant support and motivating inspiration. And we would like to dedicate this book to him in the 50th anniversary of fuzzy sets.

Adrian Ban, Lucian Coroianu and Przemysław Grzegorzewski
Oradea and Warsaw, October 2015

Chapter 1
Fuzzy numbers

1.1 Fuzzy sets

1.1.1 Modeling imprecision

Sometimes we can precisely state if a certain object belongs or not to a given set. For example, let us consider a set A of people which are 40 or less years old. If χ_A is a characteristic function of A, i.e.

$$\chi_A(x) = \begin{cases} 1 & \text{if} \quad x \in A, \\ 0 & \text{if} \quad x \notin A, \end{cases}$$

then we can say without any doubt that $\chi_A(x) = 1$ if x is a person 40 or less years old and $\chi_A(x) = 0$ if x is a person with older than 40. However, very often we cannot say for sure if an object belongs to a given set, especially if this set is described imprecisely or ambiguously, using words common in a natural language. For example, in such everyday language the opposite of the word "young" is the word "old". Therefore, the classical logic encourages us to split people in two categories: old and "young". Although in some cases it would be more or less obvious, e.g. 20 years old person is surely "young", while 80 years old man would be classified into the category of "old" persons, sometimes it would be difficult to decide into which category should belong a person 52 years old. It seems that the answer here is context dependent.

To overcome problems where the classical logic appears not sufficient to model situations under study the notion of **fuzzy set** was introduced by Lotfi Zadeh in 1965.

Definition 1.1. (Zadeh [202]) Let $\mathbb{X}$ be a universe of discourse. A **fuzzy set** A in $\mathbb{X}$ is characterized by a **membership function** $\mu_A : \mathbb{X} \to [0,1]$, which assigns to each object $x \in \mathbb{X}$ a real number in the interval $[0,1]$, so as $\mu_A(x)$ represents the degree of membership of x into A.

Keeping the notations as in the above definition, a fuzzy set A may be perceived as

$$A = \{(x, \mu_A(x)) : x \in \mathbb{X}, \mu_A(x) \in [0,1]\}.$$

The set of all fuzzy sets in $\mathbb{X}$ is denoted with $\mathbb{FS}(\mathbb{X})$. If for a fuzzy set $A \in \mathbb{FS}(\mathbb{X})$ we have $\mu_A(x) = 0$ for all $x \in \mathbb{X}$, then we say that A is an **empty set** and we write as usual $A = \emptyset$. If a set $\{x \in \mathbb{X} : \mu_A(x) > 0\}$ is finite then the corresponding fuzzy set A is called a **discrete fuzzy set**. In this we usually describe fuzzy set A by neglecting all the elements $x \in \mathbb{X}$ such that $\mu_A(x) = 0$. For instance, $A \in \mathbb{FS}(\mathbb{Z})$ given by

$$A = \{(-3, 0.2), (0, 0.5), (2, 1), (5, 0.7), (6, 0.3)\}$$

is an example of a discrete fuzzy set.

The interpretation of the grade of membership is very natural: if $\mu_A(x) = 1$ then we are sure that element x belongs to A, while in the case when $\mu_A(x) = 0$ then it surely does not belong to A. In all other cases, i.e. if $\mu_A(x) \in (0,1)$ then we have a partial membership (or partial belongingness to A). It means that if $\mu_A(x)$ is very close to 1 then the degree of membership of x in A is very high, while if $\mu_A(x)$ is very close to 0 then the degree of membership of x in A is very low. If $\mu_A(x) \in \{0,1\}$ for all $x \in \mathbb{X}$ then the fuzzy set A reduces to a set in the classical meaning. It means that each "usual" set is a fuzzy set whose membership function coincides with the characteristic function of that set. In fuzzy set theory such "usual" sets are usually called **crisp** sets.

The way we assign a degree of membership strongly depends on our perception regarding the objects that we are dealing with. It means that the particular shape of a fuzzy set might be designed in a very subjective way. Sometimes on may also apply some statistical methods for constructing membership functions (see, e.g. [70]). On the other hand in engineering, economics and other research areas where fuzzy sets are used, there are many examples of fuzzy sets with commonly accepted membership functions.

Please note, that since the membership function μ_A describes completely a corresponding fuzzy set A, many authors - to simplify and reduce the notation - denote the membership function of A by $A(x)$, instead of $\mu_A(x)$.

1.1.2 Basic operations on fuzzy sets

Basic operations on crisp sets (equality, complement, inclusion, union, intersection) can be extended in a natural way to fuzzy sets. In what follows we list the definitions of these basic operations (see Zadeh [202]). A more detailed discussion, including also other approaches, is made in Hanss [125] (see also the references cited there).

Definition 1.2. Let $A, B \in \mathbb{FS}(\mathbb{X})$.

(i) A and B are equal (and we write $A = B$) if $\mu_A(x) = \mu_B(x)$ for all $x \in \mathbb{X}$.
(ii) A is included in B (and we write $A \subseteq B$) if $\mu_A(x) \leq \mu_B(x)$ for all $x \in \mathbb{X}$.

(iii) The complement of A, denoted $\neg A(x)$, is characterized by the membership function $\mu_{\neg A(x)} : \mathbb{X} \to [0,1]$ such that

$$\mu_{\neg A}(x) = 1 - \mu_A(x),$$

for all $x \in \mathbb{X}$.

(iv) The union of A and B, denoted $A \cup B$, is characterized by the membership function $\mu_{A\cup B} : \mathbb{X} \to [0,1]$ given by

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

for all $x \in \mathbb{X}$.

(v) The intersection of A and B, denoted $A \cap B$, is characterized by the membership function $\mu_{A\cap B} : \mathbb{X} \to [0,1]$ given by

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

for all $x \in \mathbb{X}$.

The above presented operations satisfy some remarkable identities such as De Morgan's laws:

$$\neg(A \cup B) = \neg A \cap \neg B$$
$$\neg(A \cap B) = \neg A \cup \neg B,$$

associativity:

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C),$$

distributivity:

$$C \cap (A \cup B) = (C \cap A) \cup (C \cap B),$$
$$C \cup (A \cap B) = (C \cup A) \cap (C \cup B)$$

and commutativity:

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A,$$

which holds for all $A, B, C \in \mathbb{FS}(\mathbb{X})$. But they do not satisfy the law of contradiction and the law excluded middle, i.e. there exist $A \in \mathbb{FS}(\mathbb{X})$ such that $A \cap \neg A \neq \emptyset$ and $A \cup \neg A \neq \mathbb{X}$.

Apart from these classical operations on fuzzy sets, proposed by Zadeh, there are many other ways one may generalize operations defined on crisp sets into fuzzy domain. In fact we may obtain many interesting operations on fuzzy sets using so-called t-norms and t-conorms (also called s-norms). For more details we refer the reader to Section 1.4.2.

1.1.3 Height, core, support and $\alpha-$cut of a fuzzy set

The **height** of a fuzzy set $A \in \mathbb{FS}(\mathbb{X})$ is defined by

$$\mathrm{hgt}(A) = \sup_{x \in \mathbb{X}} \mu_A(x). \tag{1.1}$$

From Definition 1.1 it is immediate that for any a fuzzy set A we have $\mathrm{hgt}(A) \leq 1$. If there exists $x_0 \in \mathbb{X}$ such that $\mathrm{hgt}(A) = \mu_A(x_0) = 1$ then A is called **normal**.

The **core** of $A \in \mathbb{FS}(\mathbb{X})$ is denoted by $\mathrm{core}(A)$ and is given by

$$\mathrm{core}(A) = \{x \in \mathbb{X} : \mu_A(x) = 1\}. \tag{1.2}$$

It is immediate that $\mathrm{core}(A) \neq \emptyset$ if and only if A is normal.

The **support** of a fuzzy set $A \in \mathbb{FS}(\mathbb{X})$ is denoted by $\mathrm{supp}(A)$ and represents the set of all elements of $\mathbb{X}$ with a nonzero degree of membership, i.e.

$$\mathrm{supp}(A) = \{x \in \mathbb{X} : \mu_A(x) > 0\}. \tag{1.3}$$

It is easy to check that $A \neq \emptyset$ if and only if $\mathrm{supp}(A) \neq \emptyset$.

Another notion that plays an important role in the theory of fuzzy sets is the so-called **α-cut**. For $\alpha \in [0,1]$ the α-cut of a fuzzy set $A \in \mathbb{FS}(\mathbb{X})$, denoted A_α, is given by

$$A_\alpha = \{x \in \mathbb{X} : \mu_A(x) \geq \alpha\}. \tag{1.4}$$

It is immediate that $A_0 = \mathbb{X}$ and $A_1 = \mathrm{core}(A)$.

It is clear that knowing the membership function od a given fuzzy set we can find all its α-cuts by formula (1.4). But, what is interesting, knowing all α-cuts of a fuzzy number we can also reconstruct its membership function.

Lemma 1.1. *(see, e.g., Hanss [125], p. 20) If $A \in \mathbb{FS}(\mathbb{X})$ then*

$$\mu_A(x) = \sup_{\alpha \in [0,1]} \alpha \cdot \chi_{A_\alpha}(x), \tag{1.5}$$

for every $x \in \mathbb{X}$, where χ_{A_α} is the characteristic function of the set A_α.

The above result proves that each fuzzy set is completely determined by its $\alpha-$cuts. We illustrate this by the following example.

Example 1.1. Let $A \in \mathbb{FS}(\mathbb{R})$ such that $A_\alpha = [\alpha + 1, 5 - \alpha]$ for $\alpha \in [0,1]$. By (1.5) after some elementary calculus we obtain

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x \in (-\infty, 1] \cup [5, \infty), \\ x+1 & \text{if} \quad x \in [1, 2], \\ 1 & \text{if} \quad x \in [2, 4], \\ 5-x & \text{if} \quad x \in [4, 5]. \end{cases}$$

A membership function of the fuzzy number A is given in Figure 1.1. □

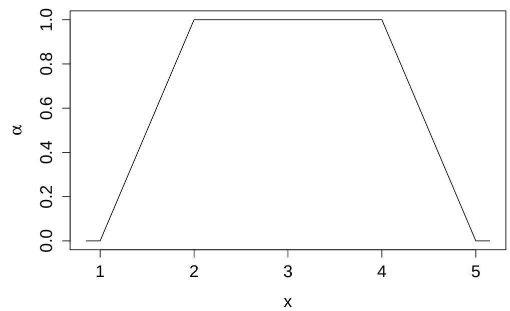

Fig. 1.1 A membership functions of the fuzzy number A (see Example 1.1).

1.1.4 Convex fuzzy sets

The notion of convexity for fuzzy sets is introduced (see Zadeh [202]) in a way which allows to preserve the properties of the ordinary convex sets. Convexity is useful both from the theoretical point of view as well as in practice, especially in pattern classification, optimization, etc.

Definition 1.3. Let $\mathbb{X}$ be a convex subset of a real vector space. We say that $A \in \mathbb{FS}(\mathbb{X})$ is **convex** if $A_\alpha = \{x \in \mathbb{X} : \mu_A(x) \geq \alpha\}$ is a convex subset of $\mathbb{X}$ for all $\alpha \in [0, 1]$.

It is immediate that $A \in \mathbb{FS}(\mathbb{X})$ is convex if and only if the membership function μ_A is a quasi-concave function, i.e.

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}, \tag{1.6}$$

for any $x_1, x_2 \in \mathbb{X}$ and $\lambda \in [0, 1]$. In many textbooks and papers formula (1.6) is used as a definition of a convex fuzzy set.

The following lemma devoted to the particular case when $\mathbb{X} = \mathbb{R}$ is useful.

Lemma 1.2. *Suppose* $A \in FS(\mathbb{R})$ *has a continuous membership function and* $\text{supp}(A)$ *is bounded. Then* A *is convex if and only if there exist* $a, b, c \in \mathbb{R}, a \leq c \leq b$ *such that*

a) $\mu_A = 0$ *outside the interval* $[a,b]$,
b) μ_A *is nondecreasing on the interval* $[a,c]$,
c) μ_A *is nonincreasing on the interval* $[c,b]$.

In this way the interpretation of a convex fuzzy set is more clear and, additionally, we can easily find examples of fuzzy sets that are non-convex.

1.1.5 Extension principle

The **extension principle**, introduced by Zadeh in [202], allows to extend basic mathematical concepts for fuzzy quantities. The n-dimensional case of Zadeh's extension principle, especially for $n = 2$, is important because it allows us to extend operations between real numbers.

Definition 1.4. (see, e.g., Hanss [125], p. 41) Let $\mathbb{X}_1, \ldots, \mathbb{X}_n$, $\mathbb{Y}$ be non-empty sets and let us consider the function $f : \mathbb{X} \to \mathbb{Y}$ where $\mathbb{X}$ is the product space $\mathbb{X} = \mathbb{X}_1 \times \ldots \times \mathbb{X}_n$. Furthermore, we consider $A_i \in \mathbb{FS}(\mathbb{X}_i)$ for all $i \in \{1, \ldots, n\}$. Using function f we can define a fuzzy set $C = f(A_1, \ldots, A_n) \in \mathbb{FS}(\mathbb{Y})$ characterized by the membership function $\mu_C : \mathbb{Y} \to [0,1]$

$$\mu_C(y) = \begin{cases} \sup\limits_{(x_1,\ldots,x_n) \in f^{-1}(y)} \min\{\mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n)\} & \text{if} \quad y \in f(\mathbb{X}), \\ 0 & \text{otherwise.} \end{cases} \tag{1.7}$$

The above formula indicates a straightforward way how to extend various operations defined on crisp sets into fuzzy environment.

Example 1.2. If $A, B \in FS(\mathbb{Z})$ are given by

$$A = \{(-1,0.2), (0,0.5), (2,1), (3,0.6), (6,0.2)\},$$
$$B = \{(-2,0.3), (-1,0.5), (0,0.8), (1,1), (3,0.7), (5,0.4)\}$$

and we consider a function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $f(x,y) = x + y$ then by (1.7) we get $C := f(A,B) = A + B$, where

$$\mu_C(z) = \max_{x+y=z} \min\{\mu_A(x), \mu_B(y)\},$$

that is

$$A + B = \{(-3,0.2), (-2,0.3), (-1,0.5), (0,0.5), (1,0.5), (2,0.8), (3,1),$$
$$(4,0.6), (5,0.7), (6,0.6), (7,0.4), (8,0.4), (9,0.2), (11,0.2)\}.$$

□

1.2 Fuzzy numbers - definitions and representations

Fuzzy numbers are fuzzy sets in $\mathbb{R}$ which satisfy some additional properties. Since they generalize real numbers they are basic for theoretical development of fuzzy set theory (fuzzy analysis, fuzzy differential equations, etc.) and very useful in numerous applications related to the representation and handling of uncertainty and incomplete information in decision making, linguistic controllers, biotechnological systems, expert systems, data mining, pattern recognition, etc.

Definition 1.5. (see [85]) A **fuzzy number** A is a fuzzy set in $\mathbb{R}$ which satisfies the following properties:

(i) A is normal,
(ii) A is convex,
(iii) μ_A is upper semicontinuous in every $x_0 \in \mathbb{R}$ (i.e. $\forall \varepsilon > 0, \exists \delta > 0$ such that $\mu_A(x) - \mu_A(x_0) < \varepsilon$, whenever $|x - x_0| < \delta$),
(iv) $cl\{x \in \mathbb{R} : \mu_A(x) > 0\}$ is bounded, where cl denotes the closure operator.

A family of all fuzzy numbers will be denoted by $\mathbb{F}(\mathbb{R})$.

Example 1.3. A fuzzy set $A \in \mathbb{FS}(\mathbb{R})$ given by

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ x^2 & \text{if} \quad 0 \le x < 2 \\ \frac{1}{2}x^2 - \frac{7}{2}x + 6 & \text{if} \quad 2 \le x < 4 \\ 0 & \text{if} \quad x \ge 4 \end{cases}$$

is a fuzzy number. Its membership function is given in Figure 1.2. □

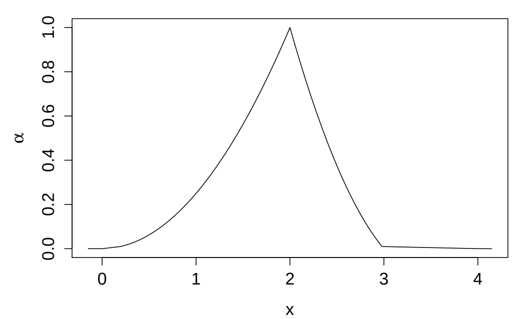

Fig. 1.2 A membership functions of the fuzzy number A (see Example 1.3).

Please note that any real number x_0 is a fuzzy number with the membership function equal to the characteristic function $\chi_{\{x_0\}}$. Similarly, any real interval $[a, b]$

is a fuzzy number with with the membership function equal to the characteristic function $\chi_{[a,b]}$.

The α-cuts A_α of a fuzzy number A are given by $A_\alpha = \{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}$ for $\alpha \in (0,1]$ and $A_0 = cl\,\{x \in \mathbb{R} : \mu_A(x) > 0\}$.

The following result, known as the Stacking Theorem, gives us important information about α-cuts.

Theorem 1.1. *(Negoiţă-Ralescu [159]) Let $A \in \mathbb{F}(\mathbb{R})$ with its α-cuts A_α, $\alpha \in [0,1]$. Then*

a) A_α is a closed interval, $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, for any $\alpha \in [0,1]$,
b) if $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ then $A_{\alpha_2} \subseteq A_{\alpha_1}$,
c) for any sequence $\{\alpha_n\}$ which converges from below to $\alpha \in (0,1]$ we have $\bigcap_{n=1}^{\infty} A_{\alpha_n} = A_\alpha$,
d) for any sequence $\{\alpha_n\}$ which converges from above to 0 we have $cl\,(\bigcup_{n=1}^{\infty} A_{\alpha_n}) = A_\alpha$.

The endpoints of each α-cut A_α, $\alpha \in [0,1]$, are given by

$$A_L(\alpha) = \inf\{x \in \mathbb{R} : \mu_A(x) \geq \alpha\} \tag{1.8}$$
$$A_U(\alpha) = \sup\{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}. \tag{1.9}$$

It is easily seen that the following definition of a fuzzy number is equivalent to Definition 1.5.

Definition 1.6. A fuzzy number A is a fuzzy set characterized by a membership function $\mu_A : \mathbb{R} \to [0,1]$ of the form

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x \leq a_1, \\ l_A(x) & \text{if} \quad a_1 \leq x \leq a_2, \\ 1 & \text{if} \quad a_2 \leq x \leq a_3, \\ r_A(x) & \text{if} \quad a_3 \leq x \leq a_4, \\ 0 & \text{if} \quad a_4 \leq x, \end{cases} \tag{1.10}$$

where $a_1, a_2, a_3, a_4 \in \mathbb{R}$, $l_A : [a_1, a_2] \longrightarrow [0,1]$ is a nondecreasing upper semicontinuous function, $l_A(a_1) = 0$, $l_A(a_2) = 1$, called the left side of the fuzzy number and $r_A : [a_3, a_4] \longrightarrow [0,1]$ is a nonincreasing upper semicontinuous function, $r_A(a_3) = 1, r_A(a_4) = 0$, called the right side of the fuzzy number.

If the sides of the fuzzy number A are strictly monotone then one can see easily that A_L and A_U are inverse functions of l_A and r_A respectively. Moreover, it can be proved that the functions A_L and A_U are left continuous.

By (1.5) we can define a fuzzy number using its α-cut representation. Consequently, we obtain the following definition of a fuzzy number introduced by Goetschel and Voxman in the paper [97].

Definition 1.7. A fuzzy number A is an ordered pair of left continuous functions $A_L, A_U : [0,1] \to \mathbb{R}$, which satisfy the following requirements:

(i) A_L is nondecreasing on $[0,1]$,
(ii) A_U is nonincreasing on $[0,1]$,
(iii) $A_L(1) \leq A_U(1)$.

If a fuzzy number A is defined by Definition 1.6 we say that A is given in *L-R* form. Otherwise, if A is defined using Definition 1.7, we say that A is given in *L-U* form.

Even if Definitions 1.6 and 1.7 are equivalent, we cannot always pass from *L-R* representation to *L-U* representation. This is easily observed since the passing from *L-R* representation to *L-U* requires the calculus of the inverses of the side functions which cannot always be performed. For this reason, in some situations such like the approximation of fuzzy numbers we need to use the same type of representation for all fuzzy numbers.

An important class of fuzzy numbers often used in practice is a family of symmetric fuzzy numbers defined as follows.

Definition 1.8. A fuzzy number A is called a **symmetric fuzzy number** if

$$A_L(1) - A_L(\alpha) = A_U(\alpha) - A_U(1),$$

for all $\alpha \in [0,1]$.

At the end of this section we will discuss about the equality of two fuzzy numbers. Due to the fact that most of the main results of the book are in relation with L_p-type metrics we adopt the following definition.

Definition 1.9. We say that fuzzy numbers A and B are equal (and we denote $A = B$) if $A_L = B_L$ and $A_U = B_U$ for almost every $\alpha \in [0,1]$.

The above definition applies only when we work with L_p-types metrics on the space of fuzzy numbers.

1.3 Basic families of fuzzy numbers

Many types of fuzzy numbers can be found in the literature. Below we mention the most important families of fuzzy numbers. Some other types, like the Gaussian fuzzy numbers or the quadratic fuzzy numbers, which are sometimes successfully applied in engineering, are considered in e.g. [125].

1.3.1 Crisp fuzzy numbers

We say that the fuzzy number A is a **crisp fuzzy number** if there exists $c \in \mathbb{R}$ such that $\mu_A(c) = 1$ and $\mu_A(x) = 0$ for all $x \in \mathbb{R} \setminus \{c\}$. It is immediate that

$A_L(\alpha) = A_U(\alpha) = c$, for every $\alpha \in [0,1]$. For simplicity, if A is a crisp fuzzy number then we usually write $A = c$. The graph of a such crisp fuzzy number is the pair $(c,1)$ which suggests the notion of singleton, the other way a crisp fuzzy number can be called.

If $B \in \mathbb{F}(\mathbb{R})$ with α-cuts $B_\alpha = [B_L(\alpha), B_U(\alpha)]$, $\alpha \in [0,1]$, and c is a crisp fuzzy number then the sum $B + c$ is a fuzzy number with α-cuts

$$(B+c)_\alpha = [B_L(\alpha) + c, B_U(\alpha) + c].$$

Crisp fuzzy numbers are the reason why real numbers are particular cases of fuzzy numbers. It is elementary to find a bijection between the set of real numbers and the set of crisp fuzzy numbers. For this reason we will make no distinction between crisp fuzzy numbers and real numbers.

1.3.2 Interval fuzzy numbers

A fuzzy number A is called an **interval fuzzy number** if there exist the reals $a, b \in \mathbb{R}$, $a \le b$, such that $\mu_A(x) = 1$ for all $x \in [a,b]$ and $\mu_A(x) = 0$ for all $x \in \mathbb{R} \setminus [a,b]$. It is immediate that $A_L(\alpha) = a$ and $A_U(\alpha) = b$, for every $\alpha \in [0,1]$. The notation $A = [a,b]$ used there suggests the terminology of interval fuzzy number. A family of all interval fuzzy numbers will be denoted by $\mathbb{F}^I$.

If $B \in \mathbb{F}(\mathbb{R})$ with α-cuts $B_\alpha = [B_L(\alpha), B_U(\alpha)]$, $\alpha \in [0,1]$, and $A = [a,b] \in \mathbb{F}^I$ then $A + B$ is a fuzzy number with α-cuts

$$(A+B)_\alpha = [B_L(\alpha) + a, B_U(\alpha) + b].$$

1.3.3 Triangular fuzzy numbers

A fuzzy number A is called a **triangular fuzzy number** if there exist $t_1 \le t_2 \le t_3$ such that

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < t_1, \\ \frac{x-t_1}{t_2-t_1} & \text{if} \quad t_1 \le x \le t_2, \\ \frac{t_3-x}{t_3-t_2} & \text{if} \quad t_2 \le x \le t_3, \\ 0 & \text{if} \quad t_3 < x. \end{cases} \tag{1.11}$$

Since by (1.11) is represented completely by those three real values t_1,t_2 and t_3, we usually denote such triangular fuzzy number by $A = (t_1, t_2, t_3)$.

It is easily seen that the α-cuts of such triangular fuzzy number are given by

$$A_\alpha = [t_1 + (t_2 - t_1)\alpha, t_3 - (t_3 - t_2)\alpha]. \tag{1.12}$$

The family of all triangular fuzzy numbers will be denoted by $\mathbb{F}^{\Delta}(\mathbb{R})$. Since crisp fuzzy numbers and interval fuzzy numbers can be regarded as particular cases of triangular fuzzy numbers, we have

$$\mathbb{R} \subset \mathbb{F}^{I} \subset \mathbb{F}^{\Delta}(\mathbb{R}).$$

1.3.4 Trapezoidal fuzzy numbers

A generalization of the triangular fuzzy number is the **trapezoidal fuzzy number**. A trapezoidal fuzzy number T is completely determined by four real parameters $t_1 \leq t_2 \leq t_3 \leq t_4$ such that

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < t_1, \\ \frac{x-t_1}{t_2-t_1} & \text{if} \quad t_1 \leq x \leq t_2, \\ 1 & \text{if} \quad t_2 \leq x \leq t_3, \\ \frac{t_4-x}{t_4-t_3} & \text{if} \quad t_3 \leq x \leq t_4, \\ 0 & \text{if} \quad t_4 < x. \end{cases} \tag{1.13}$$

We use here the notation $T = (t_1, t_2, t_3, t_4)$. When $t_2 = t_3$, T becomes a triangular fuzzy number. If $t_2 - t_1 = t_4 - t_3$ we obtain a symmetric trapezoidal fuzzy number. One can easily verify that the α-cut of such trapezoidal fuzzy number are given by

$$T_\alpha = [t_1 + (t_2 - t_1)\alpha, t_4 - (t_4 - t_3)\alpha]. \tag{1.14}$$

A family of all trapezoidal fuzzy numbers will be denoted with $\mathbb{F}^{T}(\mathbb{R})$ and a family of all symmetric trapezoidal fuzzy numbers will be denoted with $\mathbb{F}^{ST}(\mathbb{R})$. Finally, we mention that naturally we have $\mathbb{F}^{\Delta}(\mathbb{R}) \subset \mathbb{F}^{T}(\mathbb{R})$.

A family of trapezoidal fuzzy numbers is the most important subset of fuzzy sets. It is caused by the simplicity of representation which makes easier all transformations and calculations made on trapezoidal fuzzy numbers, simplifies computer applications and usually gives more intuitive and more natural interpretation. This is also the reason why the trapezoidal approximation of fuzzy numbers is a matter of great importance. The extensive study of the above mentioned approximation is given in this monograph.

1.3.5 Semi-trapezoidal fuzzy numbers

The so called parametric fuzzy numbers were introduced in the paper [158] mainly to generalize the trapezoidal approximation problem. A **parametric fuzzy number** of type (s_L, s_R) is a fuzzy number A with α-cuts $A_\alpha = [A_L(\alpha), A_U(\alpha)]$ given by

$$A_L(\alpha) = a - \sigma(1-\alpha)^{1/s_L}, \tag{1.15}$$

$$A_U(\alpha) = b + \beta(1-\alpha)^{1/s_R}, \tag{1.16}$$

where $a, b, \sigma, \beta, s_L, s_R \in \mathbb{R}$, $a \le b$, $\sigma \ge 0$, $\beta \ge 0$, $s_L > 0$, $s_R > 0$. Note that the condition $a \le b$ is imposed in order to obtain a proper parametric fuzzy number. We use the notation $A = (a, b, \sigma, \beta)_{s_L, s_R}$. A membership function of a fuzzy number $A = (2, 3, 1, 2)_{2,0.5}$ is given in Figure 1.3.

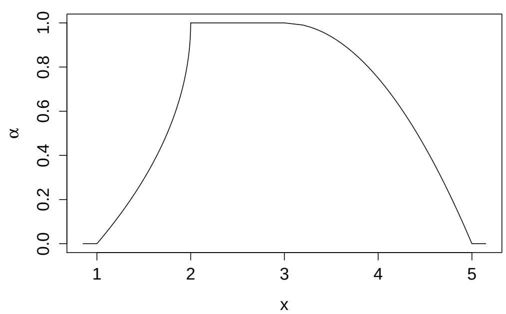

Fig. 1.3 A membership functions of a semi-trapezoidal fuzzy number $A = (2, 3, 1, 2)_{2,0.5}$.

When $s_L = s_R = 1$ then A becomes a trapezoidal fuzzy number. A family of all (s_L, s_R) fuzzy numbers will be denoted with $\mathbb{F}^{s_L, s_R}(\mathbb{R})$. Recently (see [198]) parametric fuzzy numbers are also called **semi-trapezoidal fuzzy numbers**.

1.3.6 Bodjanova type fuzzy numbers

Another important type of fuzzy numbers were introduced by Bodjanova in [48] to generalize trapezoidal fuzzy numbers. A membership of a fuzzy number from the proposed class has the following form

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < a_1, \\ \left(\frac{x-a_1}{a_2-a_1}\right)^r & \text{if} \quad a_1 \le x \le a_2, \\ 1 & \text{if} \quad a_2 \le x \le a_3, \\ \left(\frac{a_4-x}{a_4-a_3}\right)^r & \text{if} \quad a_3 \le x \le a_4, \\ 0 & \text{if} \quad a_4 < x, \end{cases} \tag{1.17}$$

where $a_1, a_2, a_3, a_4 \in \mathbb{R}$ such that $a_1 < a_2 \le a_3 < a_4$ and $r > 0$. From (1.17) and (1.13) it is easily seen that for $r = 1$ we obtain a trapezoidal fuzzy number. A Bodjanova type fuzzy number is denoted by $A = (a_1, a_2, a_3, a_4)_r$. It is immediate that

α-cuts $A_\alpha = [A_L(\alpha), A_U(\alpha)]$ of such fuzzy number are given by

$$A_L(\alpha) = a_1 + \alpha^{1/r}(a_2 - a_1),$$
$$A_U(\alpha) = a_4 - \alpha^{1/r}(a_4 - a_3).$$

A membership function of a fuzzy number $A = (-3, -1, 1, 3)_{0.4}$ is given in Figure 1.4.

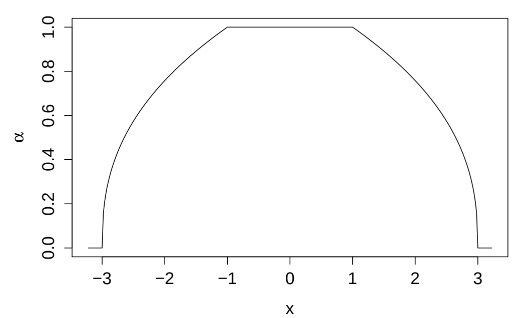

Fig. 1.4 A membership functions of a Bodjanova type fuzzy number $A = (-3, -1, 1, 3)_{0.4}$.

1.3.7 L-R fuzzy numbers

Another type of fuzzy numbers, that appears broadly in the literature, was introduced by Dubois and Prade [85] and is known as a family of *L-R* **fuzzy numbers**.

Definition 1.10. Let $L, R : [0,1] \to [0,1]$ be two continuous strictly increasing functions such that $L(0) = R(0) = 0$ and $L(1) = R(1) = 1$. Moreover, consider $t_1, t_2, t_3, t_4 \in \mathbb{R}$ such that $t_1 \le t_2 \le t_3 \le t_4$. Then a fuzzy number A given by

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < t_1, \\ L\left(\frac{x-t_1}{t_2-t_1}\right) & \text{if} \quad t_1 \le x \le t_2, \\ 1 & \text{if} \quad t_2 \le x \le t_3, \\ R\left(\frac{t_4-x}{t_4-t_3}\right) & \text{if} \quad t_3 \le x \le t_4, \\ 0 & \text{if} \quad t_4 < x, \end{cases} \tag{1.18}$$

is called an *L-R* fuzzy number.

The set of all *L-R* fuzzy numbers will be denoted by $\mathbb{F}_{L,R}(\mathbb{R})$ and an element of $\mathbb{F}_{L,R}(\mathbb{R})$ as above by $A = (t_1, t_2, t_3, t_4)_{L,R}$. The α-cuts $A_\alpha = [A_L(\alpha), A_U(\alpha)]$ of a fuzzy number (1.18) are given by

$$A_L(\alpha) = t_1 + (t_2 - t_1)L^{-1}(\alpha),$$
$$A_U(\alpha) = t_4 - (t_4 - t_3)R^{-1}(\alpha).$$

It is worth noting that if $L(x) = R(x) = x$, $x \in [0,1]$ then $\mathbb{F}_{L,R}(\mathbb{R}) = \mathbb{F}^T(\mathbb{R})$.

Example 1.4. Suppose that $L(x) = R(x) = x^3$, $x \in [0,1]$ and let $t_1 = 0$, $t_2 = 2$, $t_3 = 3$ and $t_4 = 5$. We obtain

$$\mu_A(x) = \begin{cases} \frac{x^3}{8} & \text{if} \quad 0 \le x \le 2, \\ 1 & \text{if} \quad 2 \le x \le 3, \\ \left(\frac{5-x}{2}\right)^3 & \text{if} \quad 3 \le x \le 5, \\ 0 & \text{if} \quad x \notin [0,5]. \end{cases}$$

Moreover, $A_\alpha = [2\sqrt[3]{\alpha}, 5 - \sqrt[3]{\alpha}]$ for $\alpha \in [0,1]$. A membership function of A is given in Figure 1.5. □

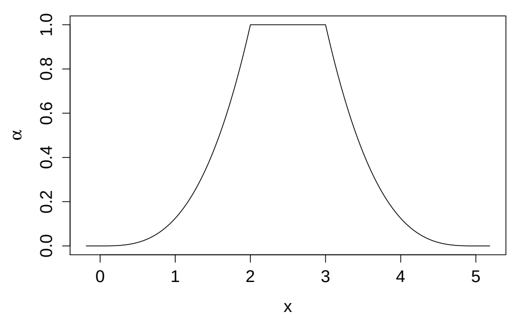

Fig. 1.5 A membership functions of the *L-R* fuzzy number *A* (see Example 1.4).

1.3.8 Piecewise linear fuzzy numbers

Trapezoidal fuzzy numbers might be generalized also by considering fuzzy numbers with piecewise linear side functions each consisting of two segments.

Definition 1.11. ([76]) An α_0**-piecewise linear** 1**-knot fuzzy number** A, where $\alpha_0 \in (0,1)$, is a fuzzy number with the following membership function

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < a_1 \\ \alpha_0 \frac{x-a_1}{a_2-a_1} & \text{if} \quad a_1 \le x < a_2 \\ \alpha_0 + (1-\alpha_0)\frac{x-a_2}{a_3-a_2} & \text{if} \quad a_2 \le x < a_3 \\ 1 & \text{if} \quad a_3 \le x \le a_4 \\ \alpha_0 + (1-\alpha_0)\frac{a_5-x}{a_5-a_4} & \text{if} \quad a_4 < x \le a_5 \\ \alpha_0 \frac{a_6-x}{a_6-a_5} & \text{if} \quad a_4 < x \le a_6 \\ 0 & \text{if} \quad x > a_6, \end{cases} \tag{1.19}$$

where $a_1 \le ... \le a_6$.

A set of all α_0-piecewise linear 1-knot fuzzy numbers will be denoted by $\mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$. An element $A \in \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R})$ as above can be denoted as $A = (\alpha_0, (a_1, a_2, a_3, a_4, a_5, a_6))$, while the α-cuts $A_\alpha = [A_L(\alpha), A_U(\alpha)]$ of fuzzy number (1.19) are given by

$$A_L(\alpha) = \begin{cases} a_1 + (a_2 - a_1)\frac{\alpha}{\alpha_0} & \text{if} \quad \alpha \in [0, \alpha_0), \\ a_2 + (a_3 - a_2)\frac{\alpha - \alpha_0}{1-\alpha_0} & \text{if} \quad \alpha \in [\alpha_0, 1], \end{cases}$$

$$A_U(\alpha) = \begin{cases} a_5 + (a_6 - a_5)\frac{\alpha_0 - \alpha}{\alpha_0} & \text{if} \quad \alpha \in [0, \alpha_0), \\ a_4 + (a_5 - a_4)\frac{1-\alpha}{1-\alpha_0} & \text{if} \quad \alpha \in [\alpha_0, 1]. \end{cases}$$

A membership function of a fuzzy number $A = (0.75, (1, 1.5, 2, 3, 4.5, 5))$ is given in Figure 1.6.

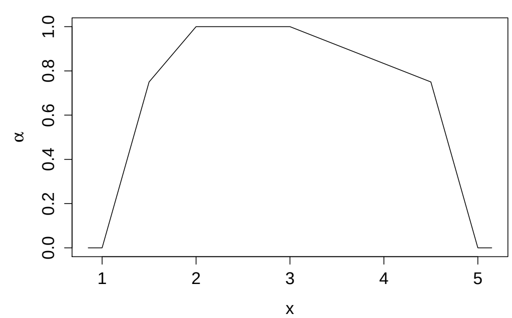

Fig. 1.6 A membership functions of the α_0-piecewise linear 1-knot fuzzy number $A = (0.75, (1, 1.5, 2, 3, 4.5, 5))$.

1.4 Operations on fuzzy numbers

1.4.1 Basic standard operations on fuzzy numbers

We start with a very important result which gives sufficient conditions for closed operations on the set of fuzzy numbers.

Theorem 1.2. *(see [42, 93, 160]) Let us consider a continuous function $f:\mathbb{R}^n \to \mathbb{R}$ and suppose that $A_1,...,A_n$ are fuzzy numbers. Then $Z = f(A_1,...,A_n)$ obtained by the extension principle (1.7) is a fuzzy number. Moreover, we have*

$$Z_\alpha = f\left((A_1)_\alpha, (A_2)_\alpha, ..., (A_n)_\alpha\right), \quad \alpha \in [0,1].$$

Note that Nguyen considered only binary operations in the above theorem but by mathematical induction the above generalization is obvious.

Since fuzzy numbers extend real numbers it would be natural to introduce basic arithmetic operations such as addition, subtraction, multiplication or division on the space of fuzzy numbers as well. The natural way is to apply the Zadeh extension principle given in Definition 1.4. It is interesting that, by considering the α-cut representation of fuzzy numbers, we get operations which are strong related with the corresponding operations in interval analysis (see, e.g., [155, 156]).

Addition of fuzzy numbers

Consider a function $f:\mathbb{R}^2 \to \mathbb{R}$ such that $f(x,y) = x+y$ and take two arbitrary fuzzy numbers A and B. We denote $f(A,B) = A+B$ and by (1.7) we obtain a fuzzy set with the following membership function

$$\begin{aligned} \mu_{A+B}(z) &= \sup_{(x,y)\in\mathbb{R}^2: x+y=z} \left(\min\{\mu_A(x), \mu_B(y)\}\right) \\ &= \sup_{x\in\mathbb{R}} \left(\min\{\mu_A(x), \mu_B(z-x)\}\right). \end{aligned} \tag{1.20}$$

We say that $A+B$ is the **sum of fuzzy numbers** A and B. Since f is continuous, by Theorem [160] it results that $A+B$ is a fuzzy number with the following α-cuts

$$\begin{aligned} (A+B)_\alpha &= A_\alpha + B_\alpha \\ &= [A_L(\alpha)+B_L(\alpha), A_U(\alpha)+B_U(\alpha)], \end{aligned} \tag{1.21}$$

for each $\alpha \in [0,1]$. Therefore, in particular we have $\mathrm{supp}(A+B) = \mathrm{supp}(A) + \mathrm{supp}(B)$ and $\mathrm{core}(A+B) = \mathrm{core}(A) + \mathrm{core}(B)$. Please note, that in (1.20) we can take max instead of sup.

The following two lemmas, although immediate to prove, are useful in practice.

Lemma 1.3. *If A and B are trapezoidal fuzzy numbers described as $A = (t_1,t_2,t_3,t_4)$ and $B = (s_1,s_2,s_3,s_4)$, respectively, then $A+B$ is also a trapezoidal fuzzy number and it could be denoted as*

$$A+B=(t_1+s_1,t_2+s_2,t_3+s_3,t_4+s_4)\,.$$

Example 1.5. Let us consider two triangular fuzzy numbers (i.e. a particular trapezoidal fuzzy numbers) $A=(1,2,3)$ and $B=(2,3,4)$. One can easily find that their sum $A+B=(3,5,7)$ is also a triangular fuzzy number. Membership functions of A, B and $A+B$ are given in Figure 1.7. □

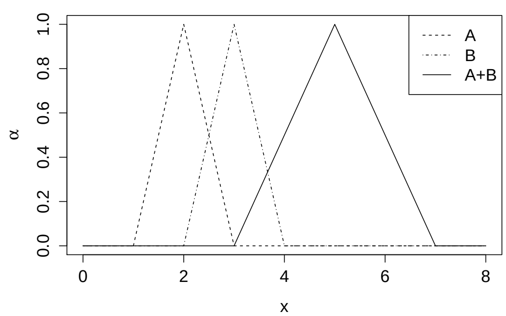

Fig. 1.7 Membership functions of A, B and $A+B$ (see Example 1.5).

Lemma 1.4. *If A and B are L-R fuzzy numbers (with fixed L and R) described as* $A=(t_1,t_2,t_3,t_4)_{L,R}$ *and* $B=(s_1,s_2,s_3,s_4)_{L,R}$*, respectively, then* $A+B$ *is also a L-R fuzzy number and it could be denoted as*

$$A+B=(t_1+s_1,t_2+s_2,t_3+s_3,t_4+s_4)_{L,R}\,.$$

Subtraction of fuzzy numbers

One may define a subtraction of two fuzzy numbers A and B similarly as their sum. Finally, we receive that $A-B$ is a a fuzzy number, called the **difference of fuzzy numbers** A and B, with the membership function

$$\mu_{A-B}(z)=\sup_{(x,y)\in\mathbb{R}^2:x-y=z}(\min\{\mu_A(x),\mu_B(y)\}) \tag{1.22}$$

and the following α-cuts

$$\begin{aligned}(A-B)_\alpha &= A_\alpha - B_\alpha &(1.23)\\ &= [A_L(\alpha)-B_U(\alpha),A_U(\alpha)-B_L(\alpha)], &(1.24)\end{aligned}$$

for each $\alpha\in[0,1]$.

Multiplication of fuzzy numbers

Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x,y) = x \cdot y$ and take two arbitrary fuzzy numbers A and B. We denote $f(A,B) = A \cdot B$ and by (1.7) we obtain a fuzzy set with the following membership function

$$\mu_{A \cdot B}(z) = \sup_{(x,y) \in \mathbb{R}^2 : x \cdot y = z} (\min\{\mu_A(x), \mu_B(y)\}) \tag{1.25}$$

We say that $A \cdot B$ is a **product of fuzzy numbers** A and B, and since f is continuous it results that $A \cdot B$ is a fuzzy number with the following α-cuts

$$\begin{aligned} A_\alpha &= A_\alpha \cdot B_\alpha \\ &= \left[\min\{A_\alpha^L B_\alpha^L, A_\alpha^L B_\alpha^U, A_\alpha^U B_\alpha^L, A_\alpha^U B_\alpha^U\},\right. \\ &\quad \left.\max\{A_\alpha^L B_\alpha^L, A_\alpha^L B_\alpha^U, A_\alpha^U B_\alpha^L, A_\alpha^U B_\alpha^U\}\right], \end{aligned} \tag{1.26}$$

for each $\alpha \in [0,1]$.

Consider the following example showing that contrary to Lemma 1.3 the product of two trapezoidal fuzzy numbers may be not a trapezoidal one.

Example 1.6. Let us consider two trapezoidal fuzzy numbers $A = (1,2,4,6)$ and $B = (2,4,4,5)$. One can easily find that their α-cuts are as follows: $A_\alpha = [1+\alpha, 6-2\alpha]$ and $B_\alpha = [2+2\alpha, 5-\alpha]$, respectively. Then

$$\begin{aligned} (A \cdot B)_\alpha &= [1+\alpha, 6-2\alpha] \cdot [2+2\alpha, 5-\alpha] \\ &= [2+3\alpha+2\alpha^2, 30-16\alpha+2\alpha^2]. \end{aligned}$$

It is clear that $A \cdot B$ is not a trapezoidal fuzzy number. □

Division of fuzzy numbers

Consider a function $f : \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$ such that $f(x,y) = x/y$, take two fuzzy numbers A and B, where B is called a nonzero fuzzy number, i.e. $0 \notin \text{supp}(B)$. We denote $f(A,B) = A/B$ and by (1.7) we obtain a fuzzy set with the following membership function

$$\mu_{A/B}(z) = \sup_{(x,y) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\}) : x/y = z} (\min\{\mu_A(x), \mu_B(y)\}). \tag{1.27}$$

We say that A/B is a **quotient of fuzzy numbers** A and B, and it is a fuzzy number with the following α-cuts

$$\begin{aligned} (A/B)_\alpha &= A_\alpha / B_\alpha \\ &= \left[\min\{A_\alpha^L/B_\alpha^L, A_\alpha^L/B_\alpha^U, A_\alpha^U/B_\alpha^L, A_\alpha^U/B_\alpha^U\},\right. \\ &\quad \left.\max\{A_\alpha^L/B_\alpha^L, A_\alpha^L/B_\alpha^U, A_\alpha^U/B_\alpha^L, A_\alpha^U/B_\alpha^U\}\right], \end{aligned} \tag{1.28}$$

for each $\alpha \in [0,1]$ and B such that $0 \notin \text{supp}(B)$.

Scalar multiplication

In the case of fuzzy numbers we distinguish not only multiplication of fuzzy numbers as defined in (1.25), but also the so called scalar multiplication, which is easily obtain from the general multiplication presented earlier. It will suffice to characterize this operations only by the α-cut representation. Therefore, if A is an arbitrary fuzzy number and $\lambda \in \mathbb{R}$ then the scalar multiplication between λ and A will be denoted $\lambda \cdot A$ and from (1.26) it results that for any $\alpha \in [0,1]$ we have

$$(\lambda \cdot A)_\alpha = \lambda A_\alpha = \begin{cases} [\lambda A_L(\alpha), \lambda A_U(\alpha)] & \text{if} \quad \lambda \geq 0, \\ [\lambda A_U(\alpha), \lambda A_L(\alpha)] & \text{if} \quad \lambda < 0. \end{cases} \tag{1.29}$$

In particular, considering a trapezoidal fuzzy number $T = (t_1, t_2, t_3, t_4)$ it results that $\lambda \cdot T = (\lambda t_1, \lambda t_2, \lambda t_3, \lambda t_4)$ if $\lambda \geq 0$ and $\lambda \cdot T = (\lambda t_4, \lambda t_3, \lambda t_2, \lambda t_1)$ if $\lambda < 0$.

Proposition 1.1. *The following equalities hold for any fuzzy $A, B, C \in \mathbb{F}(\mathbb{R})$ and any $\lambda, \beta \in \mathbb{R}$:*

a) $A + B = B + A$,
b) $(A + B) + C = A + (B + C)$,
c) $A + 0 = A$,
d) $1 \cdot A = A$,
e) $\lambda \cdot (A + B) = \lambda \cdot A + \lambda \cdot B$,
f) $\lambda \cdot (\beta \cdot A) = \beta \cdot (\lambda \cdot A) = (\lambda \beta) \cdot A$,
g) $A \cdot B = B \cdot A$,
h) $A \cdot (B \cdot C) = (A \cdot B) \cdot C$.

It is easy to check that the only fuzzy numbers having opposite elements with respect to the addition of fuzzy numbers are the crisp numbers. Actually, it is evident that in general the property $A + (-A) = 0$ does not hold for fuzzy numbers. Therefore, the triple $(\mathbb{F}(\mathbb{R}), +, \cdot)$ is not a vector space. This lack of property causes real difficulties in some practical situations such as solving fuzzy systems of equations or when we consider for example the best approximation problem. It is well known that most of the existence results concerning the best approximation problem are given in normed vector spaces. However, as it will be seen in Chapter 3, the problem of approximating fuzzy numbers by fuzzy numbers with simpler form will be reduced to approximation problems in Hilbert spaces in the case of the L_2-type metrics. But as we look on the above proposition we observe that $(\mathbb{F}(\mathbb{R}), +, \cdot)$ is a semilinear space and therefore by Theorem 5.3 it is very important to notice that there exists a vector space$(\widetilde{\mathbb{F}(\mathbb{R})}, \oplus, \odot)$ and an injective application (inclusion) $i : \mathbb{F}(\mathbb{R}) \to \widetilde{\mathbb{F}(\mathbb{R})}$ and, regarding $\mathbb{F}(\mathbb{R})$ as a subset of $\widetilde{\mathbb{F}(\mathbb{R})}$ (that is adopting the convention $i(A) = A$ for all $A \in \mathbb{F}(\mathbb{R})$) we have

$$A \oplus B = A + B,$$
$$\lambda \odot A = \lambda \cdot A,$$

for all $A, B \in \mathbb{F}(\mathbb{R})$ and $\lambda \in [0, \infty)$.

1.4.2 Interactive operations on fuzzy numbers

In this section we will generalize the extension principle which will alow us to obtain more general formulas for the addition or multiplication of fuzzy numbers. These generalizations are important in some applications in which the extension principle may not give satisfactory results. For example, by Example 1.6 we know that standard multiplication of fuzzy numbers is not a closed operation. So, we can say that multiplication is not a shape preserving operations (in this case linear shape). Interestingly, it is possible to define a new formula for the multiplication such that the linear shape is preserved (see [130]). In order to generalize the extension principle we need the notion of a triangular norm.

Definition 1.12. (see e.g [99]) A **triangular norm** (**t-norm** for short) is a function $T : [0,1] \times [0,1] \to [0,1]$ which satisfies the following properties:

(i) $T(x,1) = x$, for all $x \in [0,1]$ (identity)
(ii) $T(x,y) = T(y,x)$, for all $x,y \in [0,1]$ (commutativity)
(iii) $T(x,T(y,z)) = T(T(x,y),z)$, for all $x,y,z \in [0,1]$ (associativity)
(iv) if $x \leq u$ and $y \leq v$ then $T(x,y) \leq T(u,v)$, for all $x,y,z \in [0,1]$ (monotonicity).

If T_1, T_2 are triangular norms such that $T_1(x,y) \leq T_2(x,y)$, for all $x,y \in [0,1]$, then we say that T_1 is weaker than T_2 (or that T_2 is stronger than T_1) and we denote $T_1 \leq T_2$ (or $T_2 \geq T_1$).

Note, that if T is a triangular norm then $T(x,x) \leq T(x,1) = x$ for any $x \in [0,1]$. Therefore, for any $x \in [0,1]$ we have $T(0,x) \leq T(0,1) = 0$ and hence $T(0,x) = 0$ for all $x \in [0,1]$.

Let us consider a few examples presenting four famous t-norms.

Example 1.7. Let $T_M : [0,1] \times [0,1] \to [0,1]$ denote a function defined as follows

$$T_M(x,y) = \min\{x,y\}. \tag{1.30}$$

It is immediate that T_M is a triangular norm. Moreover, if T is an arbitrary triangular norm then $T(x,y) \leq T(x,1) = x$ and $T(x,y) = T(y,x) \leq T(y,1) = y$. Thus $T(x,y) \leq \min\{x,y\} = T_M(x,y)$ which implies that $T \leq T_M$ for any triangular norm T. For this reason T_M is called the **strongest t-norm**. □

Example 1.8. Consider $T_w : [0,1] \times [0,1] \to [0,1]$ defined as follows

$$T_w(x,y) = \begin{cases} 0 & \text{if } \max\{x,y\} < 1, \\ \min\{x,y\} & \text{otherwise.} \end{cases} \tag{1.31}$$

It is easily seen that T_w is a triangular norm. Let T be an arbitrary triangular norm and let us choose $x,y \in [0,1]$. If $\max\{x,y\} < 1$ then $T_w(x,y) = 0 \leq T(x,y)$. If $\max\{x,y\} = 1$ then $x = 1$ or $y = 1$. If $x = 1$ then $T(x,y) = T(1,y) = y \geq \min\{x,y\} =$

$T_w(x,y)$. Similarly, if $y=1$ then $T(x,y) \geq T_w(x,y)$. Therefore $T \geq T_w$ for any triangular norm T. For this reason T_w is called the **weakest t-norm**. □

Example 1.9. Define $T_L : [0,1]\times[0,1] \to [0,1]$ as follows

$$T_L(x,y) = \max\{x+y-1, 0\}. \tag{1.32}$$

Then T_L is a triangular norm known as the **Łukasiewicz t-norm**. □

Example 1.10. Define $T_P : [0,1]\times[0,1] \to [0,1]$ as follows

$$T_P(x,y) = xy. \tag{1.33}$$

Then T_P is a triangular norm known as the **product t-norm**. □

Let A and B be two arbitrary fuzzy numbers. By (1.20) and (1.30) we can write

$$\begin{aligned}(A+B)(z) &= \sup_{x+y=z} (\min\{\mu_A(x), \mu_B(y)\}) \\ &= \sup_{x+y=z} T_M(\mu_A(x), \mu_B(y))\end{aligned}$$

for any $z \in \mathbb{R}$. Replacing above T_M with an arbitrary triangular norm T we obtain a generalization of the standard addition called **T-norm based addition** (see, e.g., [94, 127, 139, 152, 153]).

We denote the T-norm based addition of A and B by $A \oplus_T B$. The membership function of the sum $A \oplus_T B$ is given for any $z \in \mathbb{R}$ as follows

$$\mu_{A\oplus_T B}(z) = \sup_{(x,y)\in\mathbb{R}^2 : x+y=z} T(\mu_A(x), \mu_B(y)). \tag{1.34}$$

It can be proved that $A \oplus_T B$ is a fuzzy number and in addition, if T is upper semicontinuous then we can take max instead of sup in (1.34).

Since $T \leq T_M$ then $A \oplus_T B \leq A+B$. This implies that $\operatorname{core}(A \oplus_T B) \subseteq \operatorname{core}(A+B)$. On the other hand, if $z \in \operatorname{core}(A+B)$ thus there exist $x_0 \in \operatorname{core}(A)$ and $y_0 \in \operatorname{core}(B)$ such that $x_0 + y_0 = z$. Hence we have

$$\begin{aligned}\mu_{A\oplus_T B}(z) &= \sup_{x+y=z} T(\mu_A(x), \mu_B(y)) \\ &\geq T(A(x_0), B(y_0)) = T(1,1) = 1.\end{aligned}$$

Therefore, we get $\operatorname{core}(A+B) \subseteq \operatorname{core}(A \oplus_T B)$. Thus finally, by the double inclusion we obtain

$$\operatorname{core}(A+B) = \operatorname{core}(A \oplus_T B). \tag{1.35}$$

Similarly, we may define the **T-norm based multiplication** (see, e.g., [130]) of A and B, denoted as $A \odot_T B$, where the membership function of the product $A \odot_T B$ is given by

$$\mu_{A \odot_T B}(z) = \sup_{(x,y)\in\mathbb{R}^2 : x\cdot y = z} T(\mu_A(x), \mu_B(y)). \tag{1.36}$$

It is worth noting that T_w is is the only triangular norm which induces a shape preserving multiplication of *L-L* fuzzy numbers (it was proved in [130]). Please notice, that in the case of addition the situation is different. Namely, the standard addition, i.e.based on the strongest triangular norm T_M, always preserves the shape of *L-R* fuzzy numbers (see Lemma 1.4). But for an arbitrary triangular norm it may not hold. Therefore, an interesting problem is to find those triangular norms which induce a shape preserving addition. Important results concerning this problem can be found in [127, 139, 153].

Let us now discuss in some sense even more general approach than operations based on triangular norms. Firstly, let us define a notion of the so called joint possibility distribution.

Definition 1.13. (see [95])Let us consider two arbitrary fuzzy numbers A and B. A function $C : \mathbb{R}^2 \to \mathbb{R}$ is called a **joint possibility distribution** of A and B if

$$\sup_{y\in\mathbb{R}} C(x,y) = \mu_A(x)$$

for all $x \in \mathbb{R}$ and

$$\sup_{x\in\mathbb{R}} C(x,y) = \mu_B(y)$$

for all $y \in \mathbb{R}$. We say that A and B are the marginal distributions of C.

If C is upper semicontinuous then we can take operator max instead of sup in the above definition.

Example 1.11. (see, e.g., [51]) Let us consider two triangular fuzzy numbers $A = (0,0,1)$ and $B = (0,1,1)$. It can be shown that a function $C : \mathbb{R}^2 \to \mathbb{R}$ given by

$$C(x,y) = (y-x)\cdot\chi_S(x,y),$$

where χ_S denotes the characteristic function of the set $S = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \leq 1, y-x \geq 0\}$, is a joint possibility distribution of A and B. □

It is worth noting that any triangular norm T generates a joint possibility distribution C_T of A and B, where

$$C_T(x,y) = T(\mu_A(x), \mu_B(y)). \tag{1.37}$$

However, there exist joint possibility distributions which cannot be generated by triangular norms.

Now we are able to define operations on fuzzy numbers based on a joint possibility distribution. Actually, an **interactive addition of** A **and** B **based on their joint possibility distribution** C is denoted $A +_C B$ and the sum is described by its membership function given as follows (see [53])

$$\mu_{A+_C B}(z) = \sup_{(x,y)\in\mathbb{R}^2 : x+y=z} C(x,y). \tag{1.38}$$

The **interactive multiplication based on a joint possibility distribution** is denoted by $A \cdot_C BA \cdot_C B$, where the product $A \cdot_C B$ is a fuzzy set with the following membership function

$$\mu_{A\cdot_C B} = \sup_{(x,y)\in\mathbb{R}^2 : x\cdot y=z} C(x,y). \tag{1.39}$$

The interactive addition and multiplication from above are obtained using the so called interactive extension principle which introduced in [53]. Moreover, it can be proved (see [53]) that both $A +_C B$ and $A \cdot_C B$ are fuzzy numbers.

We end this section by mentioning that joint possibility distributions are used in many problems of rather statistical nature (see [51, 54, 75, 95]) but also in problems concerning fuzzy arithmetic (see [50, 53, 72, 74]). For example, in paper [53] the following problem was formulated: "Let C be a joint possibility distribution with marginal distributions A and B. At what conditions does the equality $A +_C B = A + B$ hold?" It is worth mentioning that the solution, i.e. necessary and sufficient conditions for this equality were given in [72].

1.5 Distances between fuzzy numbers

There are numerous metrics defined on the space of fuzzy numbers. In this section we will list only those metrics which are suitable to our investigation on the approximation of fuzzy numbers.

One of the most popular metric is the so called **Euclidean metric** (see [101]) given by

$$d(A,B) = \left[\int_0^1 (A_L(\alpha) - B_L(\alpha))^2 \, d\alpha + \int_0^1 (A_U(\alpha) - B_U(\alpha))^2 \, d\alpha \right]^{1/2}. \tag{1.40}$$

As an application, it is immediate that for two trapezoidal fuzzy numbers $T = (t_1, t_2, t_3, t_4)$ and $T' = (t'_1, t'_2, t'_3, t'_4)$, after elementary calculus, we obtain

$$\begin{aligned} d^2(T,T') = {} & \frac{1}{3}(t_1 - t'_1)^2 + \frac{1}{3}(t_2 - t'_2)^2 + \frac{1}{3}(t_1 - t'_1)(t_2 - t'_2) \\ & + \frac{1}{3}(t_3 - t'_3)^2 + \frac{1}{3}(t_4 - t'_4)^2 + \frac{1}{3}(t_3 - t'_3)(t_4 - t'_4). \end{aligned} \tag{1.41}$$

Yeh [199] generalized metric (1.40) considering the **weighted L_2-type distance** d_λ, defined as follows

$$d_\lambda(A,B) = \left[\int_0^1 (A_L(\alpha) - B_L(\alpha))^2 \lambda_L(\alpha) d\alpha + \int_0^1 (A_U(\alpha) - B_U(\alpha))^2 \lambda_U(\alpha) d\alpha \right]^{1/2}, \tag{1.42}$$

where, in order to obtain indeed a metric, we suppose that weighting functions $\lambda_L, \lambda_U : [0,1] \to \mathbb{R}$ are strictly positive almost everywhere on $[0,1]$ and integrable. If $\lambda_L = \lambda_U$ and $\int_0^1 \lambda_L(\alpha) d\alpha = 1/2$, we rediscover the metric of Zeng and Li ([204]). Further on we use the notation $\lambda = (\lambda_L, \lambda_U)$.

More generally, considering $p \geq 1$ and a weight $\lambda = (\lambda_L, \lambda_U)$, the weighted L_p-type distance $\delta_{p,\lambda}$ is given by

$$\delta_{p,\lambda}(A,B) = \left[\int_0^1 |(A_L(\alpha) - B_L(\alpha))|^p \lambda_L(\alpha) d\alpha \int_0^1 |(A_U(\alpha) - B_U(\alpha))|^p \lambda_U(\alpha) d\alpha \right]^{1/p}. \tag{1.43}$$

If $\lambda_L(\alpha) = \lambda_U(\alpha) = 1$, $\alpha \in [0,1]$, we prefer the notation

$$d_p(A,B) = \left[\int_0^1 |(A_L(\alpha) - B_L(\alpha))|^p d\alpha + \int_0^1 |(A_U(\alpha) - B_U(\alpha))|^p d\alpha \right]^{1/p}. \tag{1.44}$$

Another class of distances between fuzzy numbers, introduced by Bertoluzza et al. [46], is given by

$$\widetilde{D}_{f,\varphi}(A,B) = \left(\int_0^1 \widetilde{D}_f^2 (A_\alpha, B_\alpha) \, d\varphi(\alpha) \right)^{1/2},$$

where

$$\widetilde{D}_f^2([a,b],[c,d]) = \int_0^1 (t\,|a-c| + (1-t)\,|b-d|)^2 \, df(t),$$

and where f is a normalized weighting measure on $[0,1]$, while function φ satisfies usually the following conditions: $\varphi(\alpha) \geq 0$ for any $\alpha \in [0,1]$, $\alpha_1 \leq \alpha_2$ implies $\varphi(\alpha_1) \leq \varphi(\alpha_2)$ and $\int_0^1 \varphi(\alpha)\, d\alpha = 1$.

Finally, let us consider the metric proposed by Trutschnig et al. [180] as follows:

$$D^*_{\Psi,\theta}(A,B) = \left(\int_0^1 \left(D^*_\theta(A_\alpha, B_\alpha)\right)^2 d\Psi(\alpha) \right)^{1/2}, \tag{1.45}$$

where $\theta \in (0,1]$, Ψ is a weighting probability measure on $[0,1]$ given by

$$\left(D^*_\theta([a,b],[c,d])\right)^2 = (\mathrm{mid}[a,b] - \mathrm{mid}[c,d])^2 + \theta(\mathrm{spr}[a,b] - \mathrm{spr}[c,d])^2, \tag{1.46}$$

while operators mid and spr correspond to the middle point of the interval under study and its spread (the half of its length), respectively, and their are defined as follows

$$\mathrm{mid}[a_1,a_2] = \frac{a_1+a_2}{2}, \tag{1.47}$$

$$\mathrm{spr}[a_1,a_2] = \frac{a_2-a_1}{2}. \tag{1.48}$$

Combining formulae (1.45)–(1.48) and substituting there α-cuts of fuzzy numbers A and B we get the following well-known formula for the Trutschnig distance

$$D^*_{\psi,\theta}(A,B) = \left(\int_0^1 \left([\mathrm{mid}A_\alpha - \mathrm{mid}B_\alpha]^2 + \theta\,[\mathrm{spr}A_\alpha - \mathrm{spr}B_\alpha]^2 \right) \psi(\alpha) d\alpha \right)^2, \tag{1.49}$$

where $\psi : [0,1] \to [0,1]$ is a weighting function. It is worth noting that if $\psi \equiv 1$ and $\theta = 1$ then such Trutschnig distance is equivalent with the Euclidean distance, i.e. $D^*_{1,1}(A,B) = \frac{1}{2}d(A,B)$.

1.6 Other notations for fuzzy numbers

Many authors examining fuzzy numbers introduce their own notation which for some specific reasons seem to be convenient in their considerations. For instance, given notation may be more easy for a particular type of fuzzy numbers (e.g. trapezoidal) or if one works with given type of metrics (say L_2-type metrics), while the same notation may appear troublesome or even inappropriate for another. In this section we present a new notation for trapezoidal fuzzy numbers introduced by Yeh in his papers [195] and [199]. Then we show a new notations introduced by Ban and Coroianu [30] which is convenient for calculations on semi-trapezoidal fuzzy numbers.

We start with notations for trapezoidal fuzzy numbers which are suitable with the Euclidean metric d as it will be seen later. One can easily verify that the α-cut of a trapezoidal fuzzy number $T = (t_1, t_2, t_3, t_4)$ can be written as follows

$$T_\alpha = [l + x(\alpha - \frac{1}{2}), u - y(\alpha - \frac{1}{2})]. \tag{1.50}$$

Therefore, by (1.14) we easily get that

$$l = \frac{t_1 + t_2}{2}, \tag{1.51}$$

$$u = \frac{t_3 + t_4}{2}, \tag{1.52}$$

$$x = t_2 - t_1, \tag{1.53}$$

$$y = t_4 - t_3 \tag{1.54}$$

or, equivalently,

$$t_1 = l - \frac{x}{2}, \tag{1.55}$$

$$t_2 = l + \frac{x}{2}, \tag{1.56}$$

$$t_3 = u - \frac{y}{2}, \tag{1.57}$$

$$t_4 = u + \frac{y}{2}. \tag{1.58}$$

Hence, a trapezoidal fuzzy number T with the α-cuts given as in (1.50) will be also denoted as $T = [l, u, x, y]$. It is easily seen that unlike the traditional notation describing a fuzzy number by four real numbers corresponding to the borders of its support and core, our new notation uses four parameters, which indicate the location of the fuzzy number (l and u) and the spread of its arms (x and y).

Now, if $T = [l, u, x, y]$ and $T' = [l', u', x', y']$ then by (1.40) and (1.50) after some simple calculations we get

$$d^2(T, T') = (l - l')^2 + (u - u')^2 + \frac{1}{12}(x - x')^2 + \frac{1}{12}(y - y')^2. \tag{1.59}$$

Clearly, the above expression may be perceived as more convenient than formula (1.41) because it shows directly that the Euclidean distance between two trapezoidal fuzzy numbers reduces to the distance between location parameters and characteristics of spread of those two fuzzy numbers. Other benefits will be seen in Chapter 3 where we investigate the approximations of fuzzy numbers by trapezoidal fuzzy numbers.

Now let us consider the space of fuzzy numbers endowed with a weighted metric d_λ given by formula (1.42). Let us introduce the following notations:

$$a = \int_0^1 \lambda_L(\alpha) d\alpha, \tag{1.60}$$

$$b = \int_0^1 \lambda_U(\alpha) d\alpha, \tag{1.61}$$

$$\omega_L = \frac{1}{a}\int_0^1 \alpha\lambda_L(\alpha)d\alpha, \tag{1.62}$$

$$\omega_U = \frac{1}{b}\int_0^1 \alpha\lambda_u(\alpha)d\alpha, \tag{1.63}$$

$$c = \int_0^1 (\alpha-\omega_L)^2\lambda_L(\alpha)d\alpha, \tag{1.64}$$

$$d = \int_0^1 (\alpha-\omega_U)^2\lambda_U(\alpha)d\alpha. \tag{1.65}$$

Next, let T be a trapezoidal fuzzy number with α-cuts given by

$$T_\alpha = [l + x(\alpha-\omega_L), u - y(\alpha-\omega_U)], \alpha \in [0,1]. \tag{1.66}$$

Such a trapezoidal fuzzy number will be denoted for simplicity by $T = [l,u,x,y]_\lambda$ (λ is a generic notation for the pair (λ_L,λ_U).

If $T = [l,u,x,y]_\lambda$ and $T' = [l',u',x',y']_\lambda$ then the weighted distance between T and T' becomes (see Proposition 2.2 in [199])

$$d_\lambda^2(T,T') = a(l-l')^2 + b(u-u')^2 + c(x-x')^2 + d(y-y')^2. \tag{1.67}$$

It is easilt seen that formula (1.59) is obtained from (1.67) by taking $\lambda_L = \lambda_U = 1$.

Now let us consider a semi-trapezoidal fuzzy number $A = (a,b,\sigma,\beta)_{s_L,s_R}$. By (1.15) and (1.16) we know that $A_L(\alpha) = a - \sigma(1-\alpha)^{1/s_L}$ and $A_U(\alpha) = b + \beta(1-\alpha)^{1/s_R}$ for $\alpha \in [0,1]$.

In what follows we introduce new notations for semi-trapezoidal fuzzy numbers. For this purpose let denote A_L and A_U in the following form

$$A_L(\alpha) = a - \sigma\frac{s_L}{s_L+1} - \sigma\left((1-\alpha)^{1/s_L} - \frac{s_L}{s_L+1}\right),$$
$$A_U(\alpha) = b + \beta\frac{s_R}{s_R+1} + \beta\left((1-\alpha)^{1/s_R} - \frac{s_R}{s_R+1}\right).$$

We obtain

$$A_L(\alpha) = l - x\left((1-\alpha)^{1/s_L} - \frac{s_L}{s_L+1}\right), \tag{1.68}$$

$$A_U(\alpha) = u + y\left((1-\alpha)^{1/s_R} - \frac{s_R}{s_R+1}\right), \tag{1.69}$$

which gives us another representation of the semi-trapezoidal fuzzy number A, where

$$l = a - \sigma \frac{s_L}{s_L+1},$$
$$x = \sigma,$$
$$u = b + \beta \frac{s_R}{s_R+1},$$
$$y = \beta.$$

Thus a semi-trapezoidal fuzzy number $A = (a,b,\sigma,\beta)_{s_L,s_R}$ could be represented equivalently as $A = [l,u,x,y]_{s_L,s_R}$. If $s_L = s_R = 1$ then we obtain the representation of a trapezoidal fuzzy number given above so the indices s_L, s_R might be omitted.

If $A = [l,u,x,y]_{s_L,s_R}$ and $B = [l',u',x',y']_{s_L,s_R}$ then the Euclidean distance between A and B becomes (see [30], Proposition 2)

$$d^2(A,B) = (l-l')^2 + (u-u')^2 + \frac{s_L}{(s_L+2)(s_L+1)^2}(x-x')^2 + \frac{s_R}{(s_R+2)(s_R+1)^2}(y-y')^2 \quad (1.70)$$

1.7 Characteristics of fuzzy numbers

Besides the membership function and α-cuts some numerical characteristics of fuzzy numbers are frequently used. They usually describe in a concise way some specific features of a fuzzy number like its location, dispersion, etc.

The **expected interval** of a fuzzy number was introduced independently by Dubois and Prade [87] and Heilpern [126]. It is the following real interval

$$EI(A) = \left[\int_0^1 A_L(\alpha)d\alpha, \int_0^1 A_U(\alpha)d\alpha \right]. \quad (1.71)$$

The expected interval is a very important characteristic of a fuzzy number having many interesting properties and useful in many situations, like defuzzification or approximation of fuzzy numbers (see, e.g., Chapter 3). Please, note that $EI(A)$ can also be regarded as a fuzzy number (more precisely, as an interval-type fuzzy number).

The middle point of the expected interval is called the **expected value** of the fuzzy number and is defined as follows

$$EV(A) = \frac{1}{2} \left[\int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha \right]. \quad (1.72)$$

The expected value of a fuzzy number A is a characteristic of location, i.e. a such a point that indicates a value which is - in some sense - typical for a fuzzy notion modeled by A (see [87, 126]). Sometimes its generalization, called **weighted expected value**, might be interesting (see [101]). It is defined by

$$EV_q(A) = (1-q)\int_0^1 A_L(\alpha)d\alpha + q\int_0^1 A_U(\alpha)d\alpha, \tag{1.73}$$

where $q \in [0,1]$. Here, by the appropriate choice of the weight q one may draw more attention to the left or right side of a fuzzy number under study.

Another characteristic of location of a fuzzy number is called just a **value of a fuzzy number** and is defined by the following formula

$$Val_s(A) = \int_0^1 s(\alpha)(A_U(\alpha) + A_L(\alpha))d\alpha, \tag{1.74}$$

where $s : [0,1] \to [0,1]$ is a nondecreasing function satisfying $s(0) = 0$ and $s(1) = 1$, called a **reducing function** (see [81]). More precisely, (1.74) is a value of A with respect to s.

The **ambiguity** of A with respect to s is

$$Amb_s(A) = \int_0^1 s(\alpha)(A_U(\alpha) - A_L(\alpha))d\alpha. \tag{1.75}$$

The ambiguity of A may be seen as the global spread of the membership function A with the reducing function s playing a weighting role. Hence ambiguity is a measure of vagueness of a fuzzy number A.

The value and ambiguity where introduced by Delgado et. all [81] to obtain a new and simple representation of fuzzy numbers (called a **canonical representation**) and to use them in decision-making. Since a value and ambiguity represent basic features of a fuzzy number, therefore according to Delgado et. all opinion two fuzzy numbers with the same ambiguity and value might be considered as equal.

If $s_k(\alpha) = \alpha^k$ for a fixed $k \in \mathbb{N}$, $\alpha \in [0,1]$, then for simplicity we denote $Val_{s_k}(A) = Val_k(A)$ and $Amb_{s_k}(A) = Amb_k(A)$ and which means that

$$Val_k(A) = \int_0^1 \alpha^k (A_U(\alpha) + A_L(\alpha))d\alpha, \tag{1.76}$$

and

$$Amb_k(A) = \int_0^1 \alpha^k (A_U(\alpha) - A_L(\alpha))d\alpha. \tag{1.77}$$

The most often used reducing function is $s_k(\alpha) = \alpha$ and hence in many paper by the value and ambiguity one simply considers $Amb_1(A) = Amb(A)$ and $Val_1(A) = Val(A)$ i.e.

$$Val(A) = \int_0^1 \alpha(A_U(\alpha) + A_L(\alpha))d\alpha \tag{1.78}$$

and

$$Amb(A) = \int_0^1 \alpha(A_U(\alpha) - A_L(\alpha))d\alpha, \tag{1.79}$$

respectively.

To describe the spread of the left-hand right-hand of a fuzzy number A with respect to the expected value, the **left-hand ambiguity** and **right-hand ambiguity** of A were introduced in [109] as follows:

$$Amb_L(A) = \int_0^1 \alpha(EV(A) - A_L(\alpha))d\alpha, \tag{1.80}$$

$$Amb_U(A) = \int_0^1 \alpha(A_U(\alpha) - EV(A))d\alpha. \tag{1.81}$$

Another useful parameter characterizing the nonspecifity of a fuzzy number is called the **width** of a fuzzy number (see [55]) and is defined by

$$w(A) = \int_0^1 (A_U(\alpha) - A_L(\alpha))d\alpha. \tag{1.82}$$

One may easily prove that for a fuzzy number A with a membership function μ_A we have

$$w(A) = \int_{-\infty}^{\infty} \mu_A(x)dx. \tag{1.83}$$

In what follows we will give an interpretation for the expected interval of a fuzzy number and also we will generalize this concept. Grzegorzewski [103] proved that for any fuzzy number A, the expected interval $EI(A)$ is the nearest (with respect to the Euclidean distance d) interval fuzzy number to A, that is

$$d(A, EI(A)) = \min_{B \in \mathbb{F}^I(\mathbb{R})} d(A, B).$$

In addition, it can be easily proved that the expected value of A is the nearest (with respect to the Euclidean distance d) crisp fuzzy number to A, i.e.

$$d(A, EV(A)) = \min_{c \in \mathbb{R}} d(A, c).$$

The above considerations suggests that in the case of a weighted L_2-type metric we should adjust the definition of the expected interval so that the interpretation would be the same.

Definition 1.14. ([37], Definition 9) Let d_λ, $\lambda = (\lambda_L, \lambda_U)$, be a weighted L_2-type metric defined on $F(\mathbb{R})$. For a fuzzy number A the interval

$$EI^\lambda(A) = \left[\frac{1}{a}\int_0^1 A_L(\alpha)\lambda_L(\alpha)d\alpha, \frac{1}{b}\int_0^1 A_U(\alpha)\lambda_U(\alpha)d\alpha\right], \tag{1.84}$$

where a and b are introduced in relations (1.60)-(1.61), is calld the **λ-weighted expected interval** of A.

By (1.60) and (1.61) we have

$$\frac{1}{a}\int_0^1 A_L(\alpha)\lambda_L(\alpha)d\alpha \le \frac{1}{a}\int_0^1 A_L(1)\lambda_L(\alpha)d\alpha = A_L(1)$$
$$\le A_U(1) = \frac{1}{b}\int_0^1 A_U(1)\lambda_U(\alpha)d\alpha \le \frac{1}{b}\int_0^1 A_U(\alpha)\lambda_U(\alpha)d\alpha,$$

and therefore $EI^\lambda(A)$ is well-defined. The **λ-weighted expected value** of A is given by

$$EV^\lambda(A) = \frac{1}{a+b}\left[a\int_0^1 A_L(\alpha)\lambda_L(\alpha)d\alpha + b\int_0^1 A_U(\alpha)\lambda_U(\alpha)d\alpha\right].$$

It can be proved that in the case of the weighted expected interval and weighted expected value we have the same interpretation with respect to the weighted metric d_λ as in the case of the usual expected interval. The extension of the weighted expected interval and of the weighted expected value for the case of extended fuzzy numbers is done in the same way as in the case of the usual ones.

Problems

1.1. Let $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x^2 + 1$ and $A \in \mathbb{FS}(\mathbb{Z})$ given by

$$A = \{(-2, 0.5), (-1, 0.4), (0, 0.7), (1, 0.6), (2, 0.3), (3, 0.6), (4, 0.8), (5, 0.7)\}.$$

Calculate $f(A)$.

1.2. Prove that a fuzzy set given by

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ x^2 & \text{if} \quad 0 \le x < 2 \\ 1 & \text{if} \quad 2 \le x < 4 \\ x^2 - \frac{21}{2}x + 27 & \text{if} \quad 4 \le x < 6 \\ 0 & \text{if} \quad x \ge 6. \end{cases}$$

is a fuzzy number.

1.3. Let A denote an arbitrary fuzzy number. Prove that for any $\alpha \in [0,1]$

$$\mu_A(A_L(\alpha)) \ge \alpha,$$
$$\mu_A(A_U(\alpha)) \ge \alpha.$$

1.4. Let A denote a continuous fuzzy number. Prove that A_L and A_U are strictly monotone and, in addition, we have

$$\mu_A(A_L(\alpha)) = \mu_A(A_U(\alpha)) = \alpha,$$

for all $\alpha \in [0,1]$.

1.5. Let us consider a fuzzy set $A \in \mathbb{FS}(\mathbb{R})$ defined as

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 2x & \text{if} \quad 0 \le x < 0.3 \\ \frac{0.4x+0.3}{0.7} & \text{if} \quad 0.3 \le x < 1 \\ 1 & \text{if} \quad 1 \le x \le 2 \\ 1.4 - 0.2x & \text{if} \quad 2 < x \le 4 \\ 3 - 0.6x & \text{if} \quad 4 < x \le 5 \\ 0 & \text{if} \quad x > 5. \end{cases}$$

Prove that A is an α_0-piecewise linear 1-knot fuzzy number. Make a graph of its membership function and find its α-cuts.

1.6. Apply Theorem 1.2 to calculate a fuzzy number $C = F(A,B)$ knowing that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by $f(x,y) = x^2 + y^2$ while $A = (1,2,3)$ and $B = (2,4,5,6)$.

1.7. Prove that

$$\int_0^1 \alpha^k A_L(\alpha) d\alpha = \frac{1}{k+1} \left(c - \int_a^c (A(x))^{k+1} dx \right)$$

and

$$\int_0^1 \alpha^k A_U(\alpha) d\alpha = \frac{1}{k+1} \left(d + \int_d^b (A(x))^{k+1} dx \right),$$

for any continuous fuzzy number A such that $\text{supp}(A) = [a,b]$ and $\text{core}(A) = [c,d]$ and for any $k \in \mathbb{N}$.

1.8. Prove that

$$EI(A) = \left[c - \int_a^c (A(x))\,dx, d + \int_d^b (A(x))\,dx \right],$$

$$Amb_k(u) = \frac{1}{k+1} \cdot \left(d - c + \int_d^b (A(x))^{k+1}\,dx + \int_a^c (A(x))^{k+1}\,dx \right)$$

and

$$Val_k(u) = \frac{1}{k+1} \cdot \left(d + c + \int_d^b (A(x))^{k+1}\,dx - \int_a^c (A(x))^{k+1}\,dx \right)$$

for every continuous fuzzy number A with $\mathrm{supp}(A) = [a,c]$ and $\mathrm{core}(A) = [d,b]$.

1.9. Let us consider a fuzzy number A such that $\mathrm{supp}(A) = [a,b]$ and $\mathrm{core}(A) = [c,d]$. If $\alpha \in \mu_A([a,c]) = Im(l_A)$ then $\mu_A(A_L(\alpha)) = \gamma$. Similarly, if $\alpha \in \mu_A([d,b]) = Im(r_A)$ then $\mu_A(A_U(\alpha)) = \alpha$.

1.10. Let A,B denote two fuzzy numbers. Then it holds that $Im(l_A) \cup Im(l_B) = Im(l_{A+B})$ and $Im(r_A) \cup Im(r_B) = Im(r_{A+B})$.

1.11. Let A denote a fuzzy number with $\mathrm{supp}(A) = [a,b]$ and $\mathrm{core}(A) = [c,d]$. If l_A is strictly increasing then A_L is continuous. Similarly, if r_A is strictly decreasing then A_U is continuous.

1.12. Let us consider $A,B \in \mathbb{F}^{\Delta}(\mathbb{R})$, $A = B = (0,0,1)$ and $C : \mathbb{R}^2 \to \mathbb{R}$, $C(x,y) = (1-x-y)\chi_T(x,y)$ where χ_T is the characteristic function of the set $T = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x+y \leq 1\}$. Prove that C is a joint possibility distribution of fuzzy numbers A and B respectively and then compute $A +_C B$ and $A \cdot_C B$, respectively.

1.13. Consider A and B as in Problem 1.12. Consider the same requirements as in Problem 1.12 for the case when $C : \mathbb{R}^2 \to \mathbb{R}$, $C(x,y) = (1-x)\chi_S(x,y)$, where χ_S is a characteristic function of the set $S = \{(x,x) \in \mathbb{R}^2 : x \in [0,1]\}$.

1.14. Now, suppose that $\mu_A(x) = \mu_B(x) = (1-x^2)\chi_{[0,1]}(x)$. Then consider a function $C : \mathbb{R}^2 \to \mathbb{R}$, $C(x,y) = (1-x^2-y^2)\chi_T(x,y)$, where χ_T is a characteristic function of the set $T = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x^2+y^2 \leq 1\}$.

1.15. Suppose that C is joint possibility distribution of fuzzy numbers A and B such that for any $\alpha \in [0,1]$ we have $C(A_L(\alpha), B_L(\alpha)) = \alpha$ and $C(A_U(\alpha), B_U(\alpha)) = \alpha$. Prove that $A + B = A +_C B$.

1.16. We say that fuzzy number A is weaker than fuzzy number B and we denote $A \leq B$ if $\mu_A(x) \leq \mu_A(x)$ for all $x \in \mathbb{R}$. Prove that if $A \leq B$ then $B_\alpha \subseteq A_\alpha$ for all $\alpha \in [0,1]$. If in addition A and B are unimodal fuzzy numbers then prove that they have the same modal value.

1.17. Consider two arbitrary fuzzy numbers A and B and suppose that their joint possibility distribution is C. Prove that $A +_C B \leq A + B$.

1.18. Consider two arbitrary fuzzy numbers A and B and consider an arbitrary triangular norm T. Prove that $A \oplus_T B \leq A + B$ and $\text{core}\,(A \oplus_T B) = \text{core}(A + B)$.

1.19. Suppose that A and B are symmetric fuzzy numbers. Prove that for any triangular norm T the T-norm based sum $A \oplus_T B$ is a symmetric fuzzy number.

1.20. Find two symmetric fuzzy numbers A and B and a joint possibility distribution C such that $A +_C B$ is not a symmetric fuzzy number.

1.21. We say that fuzzy numbers A and B have symmetrical opposite sides (see [75]) if $\mu_A(A_L(1) - x) = \mu_B(B_U(1) + x)$ and $\mu_A(A_U(1) + x) = \mu_B(B_L(1) - x)$ for all $x \in [0, \infty)$. Then suppose that T is an upper semicontinuous triangular norm strictly increasing in each argument. Prove that $A \oplus_T B$ is a symmetric fuzzy number.

1.22. Suppose that T is a triangular norm such that $T(x,x) = x$ for all $x \in [0,1]$. Prove that $T = T_M$.

1.23. Suppose that A and B are continuous fuzzy numbers with strictly monotone sides. Prove that the strongest norm T_M is the unique upper semicontinuous triangular norm such that $A + B = A +_{T_M} B$.

1.24. For some $x \in [0,1]$ and an arbitrary triangular norm T we consider

$$x_T^{(n)} = T(x,x,...,x) = T\left(x_T^{(n-1)}, x\right), n \geq 2.$$

We say that the triangular norm T is Archimedean (see, e.g., [43]) if $\lim_{n\to\infty} x_T^{(n)} = 0$ for any $x \in (0,1)$. Prove that a continuous triangular norm is Archimedean if and only if $T(x,x) < x$ for any $x \in (0,1)$.

Chapter 2
Generalized fuzzy numbers

2.1 Intuitionistic fuzzy numbers

The notions of **intuitionistic fuzzy set** [13, 14] and **interval-valued fuzzy set** [100, 131, 170, 203] were introduced as generalizations of the concept of fuzzy set (Definition 1.1). The intuitionistic fuzzy numbers and interval-valued fuzzy numbers are important to quantify an ill-known information, have appealing interpretations and can be easily employed in applications (see, e.g., [61, 62, 63, 64, 147, 161, 189, 190, 201]).

In the present section we refer to intuitionistic fuzzy numbers (see [107]), which are particular intuitionistic fuzzy sets and extensions of fuzzy numbers as well.

Definition 2.1. ([13, 14]) Let $\mathbb{X} \neq \emptyset$ be a universe of discourse. An **intuitionistic fuzzy set** in $\mathbb{X}$ is an object $A_{\langle\rangle}$ given by

$$A_{\langle\rangle} = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\},$$

where the **membership function** $\mu_A : \mathbb{X} \to [0,1]$ and the **non-membership function** $\nu_A : \mathbb{X} \to [0,1]$ satisfy the condition

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1,$$

for every $x \in \mathbb{X}$.

The values $\mu_A(x)$ and $\nu_A(x)$ represent the degree of membership and the degree of non-membership of x into $A_{\langle\rangle}$, respectively. Sometimes, for short, we denote $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle$.

Definition 2.2. An **intuitionistic fuzzy number** is an intuitionistic fuzzy set in $\mathbb{R}$, $A_{\langle\rangle} = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathbb{R}\}$, such that both μ_A and $1 - \nu_A$ are fuzzy numbers, where

$$(1 - \nu_A)(x) = 1 - \nu_A(x), \quad \forall x \in \mathbb{R}.$$

A family of all intuitionistic fuzzy numbers will be denoted by $\mathbb{F}_{\langle\rangle}(\mathbb{R})$. It is obvious that any fuzzy number $A = \{(x, \mu_A(x)) : x \in \mathbb{R}\}$ could be represented as the following intuitionistic fuzzy number: $A_{\langle\rangle} = \{\langle x, \mu_A(x), 1-\mu_A(x)\rangle : x \in \mathbb{R}\}$.

With respect to the α-cuts of the fuzzy number $1-\nu_A$ the following equalities are immediate

$$(1-\nu_A)_L(\alpha) = (\nu_A)_L(1-\alpha) \tag{2.1}$$

and

$$(1-\nu_A)_U(\alpha) = (\nu_A)_U(1-\alpha), \tag{2.2}$$

for every $\alpha \in [0,1]$.

Definition 2.3. A trapezoidal intuitionistic fuzzy number $A_{\langle\rangle}$ is an intuitionistic fuzzy number $A_{\langle\rangle} = \langle\mu_A, \nu_A\rangle$ such that $\mu_A = (\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4)$ and $1-\nu_A = (\overline{a}_1, \overline{a}_2, \overline{a}_3, \overline{a}_4)$ are trapezoidal fuzzy numbers.

If $\underline{a}_2 = \underline{a}_3$ and $\overline{a}_2 = \overline{a}_3$ in the above definition then $A_{\langle\rangle}$ is a triangular intuitionistic fuzzy number. If $\underline{a}_i = \overline{a}_i = a$, for every $i \in \{1,2,3,4\}$, then $A_{\langle\rangle}$ can be identified with the trapezoidal fuzzy number (a,a,a,a), the triangular fuzzy number (a,a,a) or the real number a.

Remark 2.1. It is immediate that $A_{\langle\rangle} = \langle\mu_A, \nu_A\rangle$, where $\mu_A = (\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4)$ and $1-\nu_A = (\overline{a}_1, \overline{a}_2, \overline{a}_3, \overline{a}_4)$, is a trapezoidal intuitionistic fuzzy number if and only if $\overline{a}_1 \leq \underline{a}_1, \overline{a}_2 \leq \underline{a}_2, \underline{a}_3 \leq \overline{a}_3$ and $\underline{a}_4 \leq \overline{a}_4$.

Example 2.1. The trapezoidal intuitionistic fuzzy number $A = \langle\mu_A, \nu_A\rangle$, such that $\mu_A = (2,4,6,8)$ and $1-\nu_A = (1,3,7,9)$, is given in Figure 2.1. □

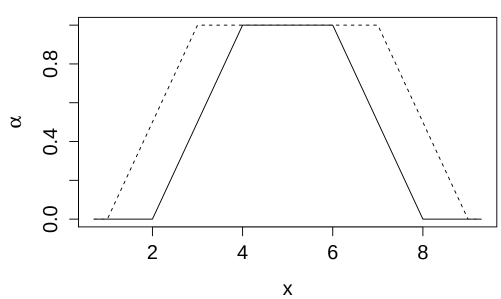

Fig. 2.1 A representation of the trapezoidal intuitionistic fuzzy number $\langle\mu_A, \nu_A\rangle$, where the solid line stands for μ_A, while the dash line for $1-\nu_A$ (see Example 2.1).

A natural extension of the weighted L_2-type distance d_λ of fuzzy numbers (1.42) to intuitionistic fuzzy numbers $A_{\langle\rangle} = \langle\mu_A, \nu_A\rangle, B_{\langle\rangle} = \langle\mu_B, \nu_B\rangle$ is defined as follows (see [16])

$$\begin{aligned}\widetilde{d_\lambda^2}(A_{\langle\rangle}, B_{\langle\rangle}) &= \frac{1}{2}\int_0^1 ((\mu_A)_L(\alpha) - (\mu_B)_L(\alpha))^2 \lambda_L(\alpha)\, d\alpha \\ &+ \frac{1}{2}\int_0^1 ((\mu_A)_U(\alpha) - (\mu_B)_U(\alpha))^2 \lambda_U(\alpha)\, d\alpha \\ &+ \frac{1}{2}\int_0^1 ((\nu_A)_L(\alpha) - (\nu_B)_L(\alpha))^2 \lambda_L(\alpha)\, d\alpha \\ &+ \frac{1}{2}\int_0^1 ((\nu_A)_U(\alpha) - (\nu_B)_U(\alpha))^2 \lambda_U(\alpha)\, d\alpha.\end{aligned}$$

It is immediate that for any $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle, B_{\langle\rangle} = \langle \mu_B, \nu_B \rangle \in \mathbb{F}_{\langle\rangle}(\mathbb{R})$

$$\widetilde{d_\lambda^2}(A_{\langle\rangle}, B_{\langle\rangle}) = \frac{1}{2} d_\lambda^2(\mu_A, \mu_B) + \frac{1}{2} d_\lambda^2(1 - \nu_A, 1 - \nu_B).$$

If $\lambda_L(\alpha) = \lambda_U(\alpha) = 1$ for every $\alpha \in [0,1]$ then a metric of Euclidean kind between intuitionistic fuzzy numbers is obtained, namely

$$\begin{aligned}d^2(A_{\langle\rangle}, B_{\langle\rangle}) &= \frac{1}{2}\int_0^1 ((\mu_A)_L(\alpha) - (\mu_B)_L(\alpha))^2 d\alpha \\ &+ \frac{1}{2}\int_0^1 ((\mu_A)_U(\alpha) - (\mu_B)_U(\alpha))^2 d\alpha \\ &+ \frac{1}{2}\int_0^1 ((\nu_A)_L(\alpha) - (\nu_B)_L(\alpha))^2 d\alpha \\ &+ \frac{1}{2}\int_0^1 ((\nu_A)_U(\alpha) - (\nu_B)_U(\alpha))^2 d\alpha.\end{aligned} \tag{2.3}$$

The operations on fuzzy numbers could be naturally extended to the family of intuitionistic fuzzy numbers. Let $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle, B_{\langle\rangle} = \langle \mu_B, \nu_B \rangle \in \mathbb{F}_{\langle\rangle}(\mathbb{R})$ and $\lambda \in \mathbb{R}$. We define the addition $A_{\langle\rangle} + B_{\langle\rangle} \in \mathbb{F}_{\langle\rangle}(\mathbb{R})$ by

$$A_{\langle\rangle} + B_{\langle\rangle} = \langle \mu_{A+B}, \nu_{A+B} \rangle,$$

where $\mu_{A+B} = \mu_A + \mu_B$ and where ν_{A+B} is given by

$$1 - \nu_{A+B} = (1 - \nu_A) + (1 - \nu_B).$$

We define the scalar multiplication $\lambda \cdot A_{\langle\rangle} \in \mathbb{F}_{\langle\rangle}(\mathbb{R})$ by

$$\lambda \cdot A_{\langle\rangle} = \langle \mu_{\lambda \cdot A}, \nu_{\lambda \cdot A} \rangle,$$

where $\mu_{\lambda \cdot A} = \lambda \cdot \mu_A$ and where $\nu_{\lambda \cdot A}$ is given by

$$1 - \nu_{\lambda \cdot A} = \lambda \cdot (1 - \nu_A).$$

Any characteristic associated with a fuzzy number can be also extended in a natural way to an intuitionistic fuzzy number $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle$ as the average of the characteristics obtained for μ_A and $1-\nu_A$. In particular, taking into account (2.1)-(2.2), the counterparts of the characteristics introduced in (1.2), (1.71), (1.72) and (1.78)-(1.82) determined for any $A = \langle \mu_A, \nu_A \rangle \in \mathbb{F}_{\langle\rangle}(\mathbb{R})$ are given as follows

$$EI\left(A_{\langle\rangle}\right) = \left[\frac{1}{2}\int_0^1 \left((\mu_A)_L(\alpha) + (\nu_A)_L(\alpha)\right) d\alpha, \right. \left. \frac{1}{2}\int_0^1 \left((\mu_A)_U(\alpha) + (\nu_A)_U(\alpha)\right) d\alpha \right], \tag{2.4}$$

$$EV\left(A_{\langle\rangle}\right) = \frac{1}{4}\int_0^1 \left((\mu_A)_L(\alpha) + (\nu_A)_L(\alpha) + (\mu_A)_U(\alpha) + (\nu_A)_U(\alpha)\right) d\alpha, \tag{2.5}$$

$$w\left(A_{\langle\rangle}\right) = \frac{1}{2}\int_0^1 \left((\mu_A)_U(\alpha) + (\nu_A)_U(\alpha) - (\mu_A)_L(\alpha) - (\nu_A)_L(\alpha)\right) d\alpha, \tag{2.6}$$

$$Val\left(A_{\langle\rangle}\right) = \frac{1}{2}\int_0^1 \alpha\,(\mu_A)_U(\alpha)d\alpha + \frac{1}{2}\int_0^1 \alpha\,(\mu_A)_L(\alpha)d\alpha + \frac{1}{2}\int_0^1 (1-\alpha)(\nu_A)_U(\alpha)d\alpha + \frac{1}{2}\int_0^1 (1-\alpha)(\nu_A)_L(\alpha)d\alpha, \tag{2.7}$$

$$Amb\left(A_{\langle\rangle}\right) = \frac{1}{2}\int_0^1 \alpha\,(\mu_A)_U(\alpha)d\alpha - \frac{1}{2}\int_0^1 \alpha\,(\mu_A)_L(\alpha)d\alpha + \frac{1}{2}\int_0^1 (1-\alpha)(\nu_A)_U(\alpha)d\alpha - \frac{1}{2}\int_0^1 (1-\alpha)(\nu_A)_L(\alpha)d\alpha, \tag{2.8}$$

$$Amb_L\left(A_{\langle\rangle}\right) = \int_0^1 \alpha \left(EV\left(A_{\langle\rangle}\right) - \frac{1}{2}(\mu_A)_L(\alpha) - \frac{1}{2}\left((1-\nu)_A\right)_L(\alpha)\right) d\alpha, \tag{2.9}$$

$$Amb_U\left(A_{\langle\rangle}\right) = \int_0^1 \alpha \left(\frac{1}{2}(\mu_A)_U(\alpha) + \frac{1}{2}\left((1-\nu)_A\right)_U(\alpha) - EV\left(A_{\langle\rangle}\right)\right) d\alpha, \tag{2.10}$$

$$core\left(A_{\langle\rangle}\right) = \left[\frac{(\mu_A)_L(1) + (\nu_A)_L(0)}{2}, \frac{(\mu_A)_U(1) + (\nu_A)_U(0)}{2}\right]. \tag{2.11}$$

2.2 Interval-valued fuzzy numbers

In this section we consider the interval-valued fuzzy numbers (see, e.g. [188]), which are extensions of fuzzy numbers and particular interval-valued fuzzy sets as well.

Definition 2.4. ([83, 128, 129]) Let $\mathbb{X} \neq \emptyset$ be a universe of discourse and $[I] = \{[a^-, a^+] : a^- \leq a^+, a^-, a^+ \in [0,1]\}$. An **interval-valued fuzzy set** in $\mathbb{X}$ is characterized by a mapping $A_{[]} : \mathbb{X} \to [I]$, which assigns to each object $x \in \mathbb{X}$ a closed interval in $[0,1]$.

Keeping the notations as in the above definition, an interval-valued fuzzy set may be perceived as

$$A_{[]} = \{(x, [\mu_{A^-}(x), \mu_{A^+}(x)]) : x \in \mathbb{X}\},$$

where the ordinary fuzzy sets $\mu_{A^-}, \mu_{A^+} : X \to [0,1]$, called the lower fuzzy set of the interval-valued fuzzy set $A_{[]}$ and the upper fuzzy set of $A_{[]}$, respectively, satisfy

$$0 \le \mu_{A^-}(x) \le \mu_{A^+}(x) \le 1$$

for any $x \in \mathbb{X}$. Sometimes, for short, we use the following notation: $A_{[]} = [A^-, A^+]$.

Definition 2.5. ([188]) An **interval-valued fuzzy number** $A_{[]}$ is an interval-valued fuzzy set in $\mathbb{R}$, $A_{[]} = \{(x, [\mu_{A^-}(x), \mu_{A^+}(x)]) : x \in \mathbb{R}\}$, which satisfies the following properties:

(i) $A_{[]}$ is normal, that is, there is an $x_0 \in \mathbb{R}$ such that $A_{[]}(x_0) = [1,1]$,
(ii) $A_{[]}$ is convex, that is, μ_{A^-} and μ_{A^+} are convex fuzzy sets,
(iii) μ_{A^-} and μ_{A^+} are upper semicontinuous,
(iv) $cl\,\{x \in \mathbb{R} : \mu_{A^-}(x) > 0\}$ and $cl\,\{x \in \mathbb{R} : \mu_{A^+}(x) > 0\}$ are bounded.

A family of all interval-valued fuzzy numbers will be denoted by $\mathbb{F}_{[]}(\mathbb{R})$. It is easily seen that the interval-valued fuzzy set

$$A_{[]} = \{(x, [\mu_{A^-}(x), \mu_{A^+}(x)]) : x \in \mathbb{R}\}$$

is an interval-valued fuzzy number if and only if A^- and A^+ are fuzzy numbers. Therefore, the lower fuzzy number A^- and the upper fuzzy number A^+ can be represented as (see 1.10)

$$A^-(x) = \begin{cases} 0 & \text{if} \quad x \le a_1^-, \\ l_{A^-}(x) & \text{if} \quad a_1^- \le x \le a_2^-, \\ 1 & \text{if} \quad a_2^- \le x \le a_3^-, \\ r_{A^-}(x) & \text{if} \quad a_3^- \le x \le a_4^-, \\ 0 & \text{if} \quad a_4^- \le x, \end{cases}$$

$$A^+(x) = \begin{cases} 0 & \text{if} \quad x \le a_1^+, \\ l_{A^+}(x) & \text{if} \quad a_1^+ \le x \le a_2^+, \\ 1 & \text{if} \quad a_2^+ \le x \le a_3^+, \\ r_{A^+}(x) & \text{if} \quad a_3^+ \le x \le a_4^+, \\ 0 & \text{if} \quad a_4^+ \le x, \end{cases}$$

where $a_1^-, a_2^-, a_3^-, a_4^-, a_1^+, a_2^+, a_3^+, a_4^+ \in \mathbb{R}$, $l_{A^-} : \left[a_1^-, a_2^-\right] \to [0,1]$, $l_{A^+} : \left[a_1^+, a_2^+\right] \to [0,1]$, $r_{A^-} : \left[a_3^-, a_4^-\right] \to [0,1]$ and $r_{A^+} : \left[a_3^+, a_4^+\right] \to [0,1]$ are nondecreasing upper semicontinuous functions,

$$l_{A^-}\left(a_1^-\right) = r_{A^-}\left(a_4^-\right) = l_{A^+}\left(a_1^+\right) = r_{A^+}\left(a_4^+\right) = 0,$$
$$l_{A^-}\left(a_2^-\right) = r_{A^-}\left(a_3^-\right) = l_{A^+}\left(a_2^+\right) = r_{A^+}\left(a_3^+\right) = 1.$$

If $a_i^- = a_i^+, i \in \{1,2,3,4\}$, $l_{A^-} = l_{A^+}$ and $r_{A^-} = r_{A^+}$ then the interval-valued fuzzy number $A_{[]}$ is a fuzzy number.

Definition 2.6. ([193]) An **interval-valued trapezoidal fuzzy number** $A_{[]}$ is an interval-valued fuzzy number $A_{[]} = [A^-, A^+]$ such that $A^- = \left(a_1^-, a_2^-, a_3^-, a_4^-\right)$ and $A^+ = \left(a_1^+, a_2^+, a_3^+, a_4^+\right)$ are trapezoidal fuzzy numbers.

If $a_2^- = a_3^-$ and $a_2^+ = a_3^+$ in Definition 2.6 then $A_{[]}$ is an **interval-valued triangular fuzzy number**. If $a_i^- = a_i^+ = a$, for every $i \in \{1,2,3,4\}$, then $A_{[]}$ can be identified with the trapezoidal fuzzy number (a,a,a,a), the triangular fuzzy number (a,a,a), or the real number a.

Remark 2.2. It is immediate that $\left[\left(a_1^-, a_2^-, a_3^-, a_4^-\right), \left(a_1^+, a_2^+, a_3^+, a_4^+\right)\right]$ is an interval-valued trapezoidal fuzzy number if and only if $a_1^+ \leq a_1^-, a_2^+ \leq a_2^-$, $a_3^- \leq a_3^+$ and $a_4^- \leq a_4^+$.

Example 2.2. The interval-valued trapezoidal fuzzy number $[(1,2,3,5),(0,1,4,7)]$ is given in Figure 2.2. □

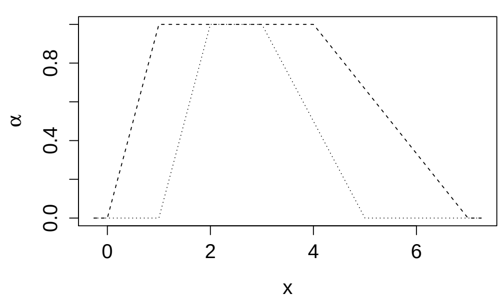

Fig. 2.2 A representation of the interval-valued trapezoidal fuzzy number $A_{[]}$, where the dot line stands for A^-, while the dash line for A^+ (see Example 2.2).

A natural extension of the weighted L_2-type distance d_λ between fuzzy numbers (see (1.42)) to interval-valued fuzzy numbers $A_{[]} = [A^-, A^+], B_{[]} = [B^-, B^+]$ is defined as follows (see [98])

$$d_\lambda\left(A_{[]}, B_{[]}\right) = \frac{1}{2}\left(\int_0^1 \left(A_L^-(\alpha) - B_L^-(\alpha)\right)^2 \lambda_L(\alpha)\, d\alpha \right.$$
$$\left. + \int_0^1 \left(A_U^-(\alpha) - B_U^-(\alpha)\right)^2 \lambda_U(\alpha)\, d\alpha \right)^{\frac{1}{2}}$$

$$+\frac{1}{2}\left(\int_0^1 \left(A_L^+(\alpha)-B_L^+(\alpha)\right)^2 \lambda_L(\alpha)\,d\alpha + \int_0^1 \left(A_U^+(\alpha)-B_U^+(\alpha)\right)^2 \lambda_U(\alpha)\,d\alpha\right)^{\frac{1}{2}}.$$

It is immediate that

$$d_\lambda\left(A_{[]},B_{[]}\right)=\frac{1}{2}d_\lambda\left(A^-,B^-\right)+\frac{1}{2}d_\lambda\left(A^+,B^+\right),$$

for every $A_{[]},B_{[]}\in\mathbb{F}_{[]}(\mathbb{R})$.

The operations on fuzzy numbers (see Section 1.4) could be extended to interval-valued fuzzy numbers. We exemplify this extension by the scalar multiplication and addition. For any $A_{[]}=[A^-,A^+]$, $B_{[]}=[B^-,B^+]$ and $\lambda\in\mathbb{R}$ we define

$$A_{[]}+B_{[]}=\left[A^-+B^-,A^++B^+\right]$$

and

$$\lambda\cdot A_{[]}=\left[\lambda\cdot A^-,\lambda\cdot A^+\right].$$

Problems

2.1. Let $A_{\langle\rangle}=\langle\mu_A,\nu_A\rangle$ and $B_{\langle\rangle}=\langle\mu_B,\nu_B\rangle$ be trapezoidal intuitionistic fuzzy numbers such that $\mu_A=(2,4,6,8)$, $1-\nu_A=(1,3,7,9)$, $\mu_B=(0,1,3,4)$ and $\nu_B=(-1,0,4,5)$. Compute $C_{\langle\rangle}=(-2)\cdot A_{\langle\rangle}+3\cdot B_{\langle\rangle}$ and $\widetilde{d_\lambda}\left(A_{\langle\rangle},B_{\langle\rangle}\right)$, where $\lambda=(\lambda_L,\lambda_U)$, $\lambda_L(\alpha)=\lambda_U(\alpha)=1,\alpha\in[0,1]$.

2.2. Let $A_{\langle\rangle}=\langle\mu_A,\nu_A\rangle$ be a trapezoidal intuitionistic fuzzy number such that $\mu_A=(2,4,6,8)$, $1-\nu_A=(1,3,7,9)$. Compute $EI\left(A_{\langle\rangle}\right)$, $EV\left(A_{\langle\rangle}\right)$, $w\left(A_{\langle\rangle}\right)$, $Val\left(A_{\langle\rangle}\right)$, $Amb\left(A_{\langle\rangle}\right)$, $Amb_L\left(A_{\langle\rangle}\right)$, $Amb_U\left(A_{\langle\rangle}\right)$, core$\left(A_{\langle\rangle}\right)$.

2.3. Let $A_{[]}$ and $B_{[]}$ be the following interval-valued trapezoidal fuzzy numbers: $A_{[]}=[(0,1,2,3),(-2,0,4,5)]$ and $B_{[]}=[(1,2,3,5),(0,1,4,7)]$. Compute $C_{[]}=(-2)\cdot A_{[]}+3\cdot B_{[]}$ and $d_\lambda\left(A_{[]},B_{[]}\right)$, where $\lambda=(\lambda_L,\lambda_U)$, $\lambda_L(\alpha)=\lambda_U(\alpha)=1,\alpha\in[0,1]$.

2.4. Let $\mu_A:\mathbb{R}\to[0,1]$ given by

$$\mu_A(x)=\begin{cases}0 & \text{if} \quad x\le 2,\\ \sqrt{x-2} & \text{if} \quad 2\le x\le 3,\\ (4-x)^2 & \text{if} \quad 3\le x\le 4,\\ 0 & \text{if} \quad x\ge 4,\end{cases}$$

and $\nu_A:\mathbb{R}\to[0,1]$ given by

$$\nu_A(x) = \begin{cases} 1 & \text{if} \quad x \leq 1, \\ 2x - x^2 & \text{if} \quad 1 \leq x \leq 2, \\ 0 & \text{if} \quad 2 \leq x \leq 3, \\ 1 - \frac{\sqrt{10-2x}}{2} & \text{if} \quad 3 \leq x \leq 5, \\ 1 & \text{if} \quad x \geq 5. \end{cases}$$

Prove that $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle$ is an intuitionistic fuzzy number.

Chapter 3
Approximations of fuzzy numbers

3.1 General remarks on approximations of fuzzy numbers

Thousands of scientific paper and practical applications proved that fuzzy set theory was recognized as an effective tool for modeling and processing imprecise information. However, sometimes membership functions representing fuzzy sets are too complicated for calculations, interpretations and further decision making. Actually, when operating with fuzzy numbers, the result of calculations strongly depend on the shape of the membership functions of these numbers. i.e. less regular membership functions lead to more complicated calculations. Moreover, fuzzy numbers with simpler shape of membership functions often have more intuitive and more natural interpretation. Hence some approximations of underlying fuzzy numbers are necessary.

The biggest simplification is realized via defuzzification, where a fuzzy number is reduced to a single point on the real line. Unfortunately, although defuzzification leads to simple structures it results in overmuch loss of information. Therefore, interval approximation of a fuzzy set is often advisable (see, e.g. [55, 103, 104, 105, 113, 167, 168]). In this approach we substitute a given fuzzy set by interval, which is - in some sense - close to the former one. Interval approximation is usually simple, effective and well-understood especially in engineering applications. However, in this type of approximation we still abandon this fundamental achievement of fuzzy set theory related to gradual distinction between objects belonging and not belonging to the considered set. Indeed, both defuzzification and interval approximation divide elements of the universe of discourse into two disjoint sets of these that belong and those that do not belong to given set.

Thus to perform an approximation that keeps gradual membership we have to look for families of fuzzy numbers with a relatively simple shape but such their support do not reduces to their core. Therefore, fuzzy numbers having linear sides appear immediately as obvious desired candidates. Of course, one may also consider fuzzy number with nonlinear sides but for the sake of simplicity the trapezoidal or triangular fuzzy numbers are most common in current applications. As noted by

Trillas: “the problems that arise with vague predicates are less concerned with precision and are more of a qualitative type; thus they are generally written as linearly as possible. Normally it is sufficient to use a trapezoidal representation, as it makes it possible to define them with no more than four parameters” (see [132]). Some justifications for restricting our attention to fuzzy numbers with linear sides are also given in [162]. Below, in further sections, we discuss some basic approaches to trapezoidal approximation which is at present a dominating topic in fuzzy number approximation.

However, before we consider all these methods in details let us look on trapezoidal approximation from different perspectives. Firstly, we may distinguish the following hierarchy of the simplification methods: defuzzification substituting a fuzzy number by a single point on the real line could be perceived as an “approximation of the first kind”. Next, the interval approximation replacing a fuzzy number by an interval which is completely characterized by two points on the real line, could be considered as an “approximation of the second kind ”. Then simplifying of a fuzzy number by a triangular fuzzy number lead to “approximation of the third kind”and finally, trapezoidal approximation might be treated as an “approximation of the fourth kind” because arbitrary fuzzy number is reduced to a trapezoidal fuzzy number which is completely characterized by four points on the real line.

Moreover, trapezoidal approximation could be also perceived as a defuzzification step considered in approximate reasoning. Roventa and Spircu [168] proposed to decompose the defuzzification process in two steps: first replacing a fuzzy set by a crisp set and then replacing the obtained crisp set by a single value. Ma et al. [149] also considered two-steps defuzzification via triangular fuzzy numbers. In the spirit of their paper one may treat trapezoidal approximation as a first step of a three-stepped defuzzification procedure. Of course, one may ask why to introduce such a sophisticated form of approximation instead of a direct defuzzification. However, it is widely known that performing defuzzification to early we lose to much information and it is better to process fuzzy information as long as possible. This is the case why we are looking for simplification to avoid difficulties in computation on the one hand and we do not want to simplify to much on the other hand. It seems that trapezoidal approximation is a reasonable compromise between these two opposite tendencies.

This chapter is organized as follows: firstly we discuss some basic properties that any desired approximation operator used for fuzzy numbers should possess. Then we describe basic methods of the interval approximation. Later we consider various approaches of the trapezoidal approximation of fuzzy numbers. Finally we mention briefly some other techniques used for simplifying fuzzy numbers.

3.2 Approximation criteria

Suppose we want to approximate a fuzzy number $A \in \mathbb{F}(\mathbb{R})$ by another fuzzy number belonging to fixed subfamily of fuzzy numbers, say $\mathbb{FN} \subset \mathbb{F}(\mathbb{R})$. In this section we adopt this general notation $\mathbb{FN}$ replaced in subsequent sections by $\mathbb{F}^I$ in the case of the interval approximation (since $\mathbb{F}^I$ is isomorphic with the family of all closed intervals on the real line), or by $\mathbb{F}^{\Delta}(\mathbb{R})$ for triangular approximation, or by $\mathbb{F}^T(\mathbb{R})$ for the trapezoidal approximation, etc. Thus we need an operator $T : \mathbb{F}(\mathbb{R}) \to \mathbb{FN}$ which transforms a family of all fuzzy numbers into a assumed subfamily of fuzzy numbers, i.e. $T : A \longmapsto T(A)$.

It is clear that whatever subfamily of fuzzy numbers $\mathbb{FN}$ we choose there are infinitely many approximations methods. Hence immediately a natural question arises: how to construct a "good" approximation operator? A similar problem appears, e.g. in statistics, if we want to indicate a good estimator of a parameter under study. And we know that there is no best estimator in a general sense but everything depends on the specified criteria and requirements. In estimation theory we have such desired criteria as consistency, unbiasedness, efficiency, etc. (see, e.g. [146]). Moreover, we face the same problems in defuzzification (here some criteria for a "good" defuzzification operator are given in [182, 169]).

Therefore, since any kind of a fuzzy number approximation could be also performed in many ways we propose a number of criteria which the approximation operator should or just can possess. There are, of course, such criteria which are desired for some specific subfamilies of fuzzy numbers $\mathbb{FN}$, which is not necessarily longer valid for another type of $\mathbb{FN}$ (e.g. if $\mathbb{FN} = \mathbb{F}^I$ we refer the reader to [55, 103, 104, 105], while if $\mathbb{FN} = \mathbb{F}^T(\mathbb{R})$ then see [116, 117, 118]). However, below we list some requirements that might be considered in a general case.

Definition 3.1. Let $d : \mathbb{F}(\mathbb{R}) \to [0,+\infty)$ denote a metric defined in the family of all fuzzy numbers. We say that an approximation operator T is **continuous** if for any $A \in \mathbb{F}(\mathbb{R})$

$$\forall(\varepsilon > 0)\ \exists(\delta > 0) \quad d(A,B) < \delta \Rightarrow d(T(A),T(B)) < \varepsilon. \tag{3.1}$$

The continuity constraint means if two original fuzzy numbers are close (in some sense) then their approximations should also be close. Or, in other words, that a small deviations in the degree of membership function should not result in a big change in the approximation. The continuity criterion is of extreme importance and hence discontinuous approximation operators seem unnatural.

Definition 3.2. We say that an approximation operator T is **invariant to translations** if for any $A \in \mathbb{F}(\mathbb{R})$

$$T(A+z) = T(A) + z \quad \forall z \in \mathbb{R}. \tag{3.2}$$

Thus translation invariance means that the relative position of the approximation remains constant when the membership function is moved to the left or to the right.

Definition 3.3. We say that an approximation operator T is **scale invariant** if for any $A \in \mathbb{F}(\mathbb{R})$

$$T(\lambda \cdot A) = \lambda \cdot T(A) \quad \forall \lambda \in \mathbb{R} \setminus \{0\}. \tag{3.3}$$

It is worth noting that for $\lambda = -1$ we get, so called, symmetry constraint, which means that the relative position of the approximation does not vary if the orientation of the support interval changes.

Definition 3.4. We say that an approximation operator T is **monotonic** if for any $A, B \in \mathbb{F}(\mathbb{R})$

$$\text{if} \quad A \subseteq B \quad \text{then} \quad T(A) \subseteq T(B). \tag{3.4}$$

Thus the monotonicity criterion states that inclusion remains invariant under approximation.

Definition 3.5. We say that an approximation operator $T : \mathbb{F}(\mathbb{R}) \to \mathbb{FN}$ satisfies the **identity criterion** if

$$T(A) = A \quad \forall A \in \mathbb{FN}. \tag{3.5}$$

This criterion states that, e.g. a reasonable trapezoidal approximation of a trapezoidal fuzzy number is equivalent to that fuzzy number.

Definition 3.6. We say that an approximation operator $T : \mathbb{F}(\mathbb{R}) \to \mathbb{FN}$ satisfies the **nearness criterion** with respect to metric d if for any $A \in \mathbb{F}(\mathbb{R})$

$$d(A, T(A)) \leq d(A, B) \quad \forall B \in \mathbb{FN}. \tag{3.6}$$

In other words, T fulfills the nearness criterion if for any fuzzy number A its output $T(A)$ is the nearest fuzzy number to A with respect to metric d among all fuzzy numbers in given subfamily $\mathbb{FN}$.

Several interesting characteristics of fuzzy numbers discussed in Section 1.7. Some of them, like the expected interval, play a key role in fuzzy number analysis and applications. This is the reason that the invariance of given characteristic of fuzzy number might be perceived as a desired property during approximation.

Definition 3.7. Let Θ denote a (real or fuzzy) characteristic of a fuzzy number. We say that an approximation operator T **preserves** Θ if for any $A \in \mathbb{F}(\mathbb{R})$

$$\Theta(T(A)) = \Theta(A). \tag{3.7}$$

Thus, in particular, we say that operator T fulfills the **expected interval invariance** if it preserves the expected interval of a fuzzy number, i.e.

$$EI(T(A)) = EI(A). \tag{3.8}$$

Similarly, we may consider other invariance criteria connected with such characteristics like the expected value EV, value Val , ambiguity Amb, width w, etc., just by substituting Θ in (3.7) by given characteristic. As a particular case of Definition 3.7 we also obtain the $\alpha-$**cut invariance**.

Definition 3.8. Let α_0 denote any fixed value in the unit interval $[0,1]$. We say that an approximation operator T is α_0-**invariant** if for any $A \in \mathbb{F}(\mathbb{R})$

$$(T(A))_{\alpha_0} = A_{\alpha_0}. \tag{3.9}$$

Three such operators are of special interest – for $\alpha = 0$, $\alpha = 0.5$ and $\alpha = 1$. Indeed, $\alpha = 0$ – invariant operator preserves the support of a fuzzy number A, i.e.

$$\mathrm{supp}(T(A)) = \mathrm{supp}(A). \tag{3.10}$$

Similarly, $\alpha = 1$ – invariant operator preserves the core of a fuzzy number A, i.e.

$$\mathrm{core}(T(A)) = \mathrm{core}(A), \tag{3.11}$$

while $\alpha = 0.5$ – invariant operator preserves a set of values that belong to A to the same extent as they belong to its complement $\neg A$.

When we approximate one model with another one, this basically means that we want to replace one type of information with an equal amount of information of another type. In other words we want to convert uncertainty of one type to another while, at the same time, preserving its amount. This expresses the spirit of, so called, the principle of uncertainty invariance (see [138]). Therefore, it seems desirable that the output of approximation $T(A)$ of a fuzzy number A should contain the same (or at least similar) amount of uncertainty as the initial fuzzy number A. Here one can consider different measures of uncertainty, like: specificity or nonspecificity, etc. (see [138]).

Definition 3.9. Let $I(A)$ denote a measure of information delivered by a fuzzy number A. We say that an approximation operator T is **preserves information** (or is **information invariant** with respect to I) if for any $A \in \mathbb{F}(\mathbb{R})$

$$I(A) = I(T(A)). \tag{3.12}$$

There is often a certain degree of vagueness in estimating a membership function. Thus, the next criterion of compatibility with the extension principle states that computing an operation on the approximation of fuzzy numbers using the extension principle will provide a result close to the result of applying this extended operation to the fuzzy numbers.

Definition 3.10. Let $d : \mathbb{F}(\mathbb{R}) \to [0, +\infty)$ denote a metric defined in the family of all fuzzy numbers. We say that an approximation operator T is **compatible with the extension principle** if for any $A \in \mathbb{F}(\mathbb{R})$ and for any operation based on the extension principle $* : \mathbb{F}(\mathbb{R}) \to \mathbb{F}(\mathbb{R})$ we have

$$\text{if} \quad d(A, T(A)) < \varepsilon, \quad d(B, T(B)) < \varepsilon \quad \text{then} \quad d(A * B, T(A) * T(B)) < O^*(\varepsilon), \tag{3.13}$$

$O^*(\varepsilon)$ is vanishing when $\varepsilon \to 0$ and is also a function of the operation $*$.

Since all fuzzy arithmetic operations are of that type, this criterion enables operations to be performed with assurance that the validity of the final approximate result will be at least equal to the validity of the intermediate approximation made.

Everybody knows that there is no unique and natural order in a family of all fuzzy numbers. Several authors have proposed different methods for ranking fuzzy numbers (see [49], [205] for a review of methods). No ranking method is the best one and it is generally chosen with respect to particular applications (we refer the reader to Chapter 4 for the detailed discussion). However, a reasonable approximation operator should preserve the accepted ordering.

Definition 3.11. Let $A \succ B$ means that A is "greater" than B with respect to ordering $\succ$. Then we say that an approximation operator T is **order invariant** with respect to $\succ$ if for any $A \in \mathbb{F}(\mathbb{R})$

$$A \succ B \Leftrightarrow T(A) \succ T(B). \tag{3.14}$$

In some applications we are interested in finding the correlation between fuzzy numbers which describes the relationship between these fuzzy numbers.

Definition 3.12. We say that an approximation operator T is **correlation invariant** if for any $A \in \mathbb{F}(\mathbb{R})$

$$\rho(A,B) = \rho(T(A), T(B)), \tag{3.15}$$

where $\rho(A,B)$ denotes a correlation of two fuzzy numbers.

Now, in the successive sections we will consider various approximation operators which are in some sense reasonable and useful in applications. They usually possess more or less those desired properties discussed above. But sometimes they are designed in such way that some other requirements, not mentioned in this section, have to be fulfilled.

3.3 Interval approximations of fuzzy numbers

3.3.1 Interval approximation without constraints

Suppose we want to approximate a fuzzy number by a crisp interval. Thus we have to use an operator $T : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^I$ which transforms fuzzy numbers into the family of all closed intervals on the real line (isomorphic with the family of all interval fuzzy numbers). Different methods for finding interval approximations of fuzzy sets are used. The easiest way is to substitute a fuzzy number either by its support

$$T_0(A) = \text{supp}(A) \tag{3.16}$$

or by its core

$$T_1(A) = \text{core}(A), \tag{3.17}$$

but using this methods all information due to fuzziness of the notion under discussion is neglected. Hence probably the best known and the most popular in practice operator is

$$T_{0.5}(A) = \{x \in \mathbb{R} : \mu_A(x) \geq 0.5\} = A_{0.5}. \tag{3.18}$$

This operator seems to be a compromise between two extremes T_0 and T_1. Moreover, it has a quite natural interpretation: any $x \in \mathbb{R}$ belongs to the approximation interval $T_{0.5}(A)$ of a fuzzy number A if and only if its degree of belongingness to A is not smaller than its degree of belongingness to the complement of A (i.e. $\neg A$). In literature operator (3.18) is sometimes called the **nearest ordinary set** of a fuzzy set A. However, this simple and natural operator has a very unpleasant drawback – the lack of continuity (see, e.g. [103]).

Last three methods, i.e. (3.16)-(3.18) are particular cases of the general $\alpha-$cut method for obtaining interval approximations of fuzzy numbers

$$T_\alpha(A) = \{x \in \mathbb{R} : \mu_A(x) \geq \alpha\} = A_\alpha, \quad \alpha \in (0,1], \tag{3.19}$$

i.e. we substitute given fuzzy number by its $\alpha-$cut, where α may be interpreted as a degree of conviction or acceptability of the imprecise information. Unfortunately all T_α operators reveal the lack of continuity.

Thus, looking for the interval approximation having some desired properties, Grzegorzewski [103] tried to find the operator $T_G : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^I$ which produces for any $A \in \mathbb{F}(\mathbb{R})$ the interval nearest to A with respect to the popular Euclidean metric (1.40), i.e.

$$T_G(A) = \arg\max_{I \in \mathbb{F}^I} d(I,A). \tag{3.20}$$

In other words, assuming $I = [a,b]$, we seek the solution of the minimization problem (in variables a,b)

$$d([a,b],A) \to \min \tag{3.21}$$

subject to $a \leq b$. It is clear that it suffices to minimize function $D(a,b) = d^2([a,b],A)$, i.e. to find partial derivatives

$$\frac{\partial D(a,b)}{\partial a} = -2\int_0^1 (A_L(\alpha) - a)d\alpha = -2\int_0^1 A_L(\alpha)d\alpha + 2a \tag{3.22}$$

$$\frac{\partial D(a,b)}{\partial b} = -2\int_0^1 (A_U(\alpha) - b)d\alpha = -2\int_0^1 A_U(\alpha)d\alpha + 2b, \tag{3.23}$$

equate them with 0, solve the system of equations and check whether we have actually obtained maximum. Finally, we obtain the following solution that the interval $I = [a,b]$ nearest to a given fuzzy number A is given by

$$T_G(A) = \left[\int_0^1 A_L(\alpha)d\alpha, \int_0^1 A_U(\alpha)d\alpha\right]. \tag{3.24}$$

Now, comparing our result (3.24) with (1.71) we obtain the following interesting corollary.

Corollary 3.1. *([103]) The real interval nearest to a given fuzzy number A with respect to the Euclidean metric d is its expected interval, i.e.* $T_G(A) = EI(A)$.

Operator T_G possesses many desired properties: it is continuous, invariant to translation and scale, it preserves the expected interval (by the definition), the expected value and the width, etc. It also fulfills two natural requirements

$$T_G(A) \subseteq \mathrm{supp}(A) \tag{3.25}$$

$$\mathrm{core}(A) \subseteq T_G(A) \tag{3.26}$$

stating that those real values that surely do not belong to A do not appear in its approximation (see (3.25)) and that none of the points that surely belong to A will be omitted (see (3.26)).

As it was mentioned above, the main drawback of the commonly used approximation operator (3.18) is its lack of continuity which may cause problems when the approximations of two similar fuzzy numbers differ significantly (see [103], Example 1). However, in particular cases T_G may reduce to $T_{0.5}$ operator. In particular, the following theorem might be proved.

Theorem 3.1. *([103]) In the subfamily of trapezoidal fuzzy numbers the interval approximation operators T_G and $T_{0.5}$ are equivalent, i.e.*

$$T_G(A) = T_{0.5}(A) \quad \forall A \in \mathbb{F}^T(\mathbb{R}). \tag{3.27}$$

The interval approximation operator T_G satisfies the nearness criterion with respect to the Euclidean distance just by its construction. However, it is worth noting that $T_G(A)$ is also the nearest to A with respect to the Hamming distance among all the intervals of the same width (see [55]).

Here a natural question arises: What happens if we consider different metrics instead of the Euclidean one in (3.21)? Let us firstly consider metric d_λ given by (1.42) with arbitrary chosen weight $\lambda = (\lambda_L, \lambda_U)$. Thus now we want to find such approximation operator T_{d_λ} that

$$T_{d_\lambda}(A) = \arg\max_{I \in \mathbb{F}^I} d_\lambda(I, A).$$

It is rather not surprising that the interval approximation operator T_{d_λ} satisfying the nearness criterion with respect to the weighted L_2-type metric (1.42), which is a direct generalization of the Euclidean distance (1.40), is a simple generalization of operator (3.24) including weights. Indeed, after some calculations we obtain

$$T_{d_\lambda}(A) = \left[\frac{1}{\int_0^1 \lambda_L(\alpha)d\alpha} \int_0^1 A_L(\alpha)\lambda_L(\alpha)d\alpha, \frac{1}{\int_0^1 \lambda_U(\alpha)d\alpha} \int_0^1 A_U(\alpha)\lambda_U(\alpha)d\alpha \right], \tag{3.28}$$

and comparing our result with (1.84) we obtain the following corollary.

Corollary 3.2. *The real interval nearest to a given fuzzy number A with respect to the weighted L_2-type metric* (1.42) *with the weight λ is its λ-weighted expected interval, i.e. $T_{d_\lambda}(A) = EI^\lambda(A)$.*

Another interesting distance we take into account is the Trutschnig distance (1.49), quite often used especially in fuzzy statistics. Our problem now is to find such approximation operator $T_{D^*_{\psi,\theta}}$ that

$$T_{D^*_{\psi,\theta}}(A) = \arg\max_{I\in\mathbb{F}^I} D^*_{\psi,\theta}(I,A).$$

A similar reasoning as given before leads us to the following solution (see [113])

$$T_{D^*_{\psi,\theta}}(A) = \left[\frac{1}{\int_0^1 \psi(\alpha)d\alpha}\int_0^1 A_L(\alpha)\psi(\alpha)d\alpha, \frac{1}{\int_0^1 \psi(\alpha)d\alpha}\int_0^1 A_U(\alpha)\psi(\alpha)d\alpha\right]. \tag{3.29}$$

Looking on (3.29) we get immediately the following conclusion:

Corollary 3.3. *([113]) For any parameter $\theta \in (0,1]$ the interval approximation operator $T_{D^*_{\psi,\theta}} : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^I$ producing intervals nearest to the input with respect to Trutschnig distance* (1.49) *does not depend on θ.*

In other words, the interval approximation of a fuzzy number we obtain using this operator does not depend on the parameter indicating the relative importance of the spreads against the mids. It means that the interval approximation operator (3.29) may be no longer denoted as $T_{D^*_{\psi,\theta}}$ but as $T_{D^*_{\psi}}$.

It can be proved (see [113]) that the interval approximation operator (3.29) is invariant to translations and scale, is monotonic and fulfills the identity criterion, is continuous and preserves some important characteristics like the expected value, value and ambiguity.

One may also ask about the relationship between $T_{D^*_{1,\theta}}$ and other interval approximation operators discussed in this section. In particular, by (3.29) and (3.28) we get the following conclusion.

Corollary 3.4. *The interval approximation operator $T_{D^*_{\psi,\theta}}$ coincides with the interval operator T_{d_λ} nearest to a given fuzzy number A with respect to the weighted L_2-type metric* (1.42) *with equal weight for both sides, i.e. $\lambda_L(\alpha) = \lambda_U(\alpha) = \psi(\alpha)$.*

The following theorem could be also proved.

Theorem 3.2. *([113]) Let $T_{D^*_{1,\theta}} : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^I$ denote an interval approximating operator producing intervals nearest to the input with respect to metric $D^*_{\psi,\theta}$ with equally weighted α-cuts, i.e. $\psi(\alpha) = 1$ for each $\alpha \in (0,1]$. Then the interval approximation operator $T_{D^*_{1,\theta}}$ is equivalent to operator* (3.24)*, i.e. $T_{D^*_{1,\theta}} = T_G$.*

Moreover, we get interesting results if the weighting function ψ is linear. The following theorem holds.

Theorem 3.3. *([113]) Let $T_{D^*_{\alpha,\theta}}$ denote the interval approximation operator* (3.29) *for the linear weighting function $\psi(\alpha) = \alpha$. Then for any fuzzy number A its interval approximation $T_G(A)$ contains $T_{D^*_{\alpha,\theta}}(A)$, i.e. $T_{D^*_{\alpha,\theta}}(A) \subseteq T_G(A)$.*

We can also prove a relationship between approximation operators similar to those given in Theorem 3.1

Theorem 3.4. *([113]) Let $T_{D^*_{\alpha,\theta}}$ denote the interval approximation operator* (3.29) *for the linear weighting function $\psi(\alpha) = \alpha$. Then in the subfamily of trapezoidal fuzzy numbers the interval approximation operator $T_{D^*_{\alpha,\theta}}$ is equivalent to the interval approximation operator* (3.19) *for $\alpha = \frac{2}{3}$, i.e.*

$$T_{D^*_{\alpha,\theta}}(A) = T_{2/3}(A) \quad \forall A \in \mathbb{F}^T(\mathbb{R}). \tag{3.30}$$

3.3.2 Interval approximation preserving ambiguity

In previous section we considered the interval approximation of fuzzy numbers under the only requirement of being the nearest to the input with respect to a given metric. Here we discuss a situation with additional invariance assumption required together with the nearness criterion.

Because a real interval $I = [a,b]$ can be represented as a fuzzy number with the α-cuts $I_\alpha = [a,b]$ for every $\alpha \in [0,1]$, by (1.79) we get

$$Amb([a,b]) = \frac{b-a}{2}.$$

Suppose now we are looking for the operator $T_{amb} : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^I$ which produces for any $A \in \mathbb{F}(\mathbb{R})$ the interval nearest to A with respect to the popular Euclidean metric (1.40), which preserves, additionally, the ambiguity of A, i.e.

$$T_{amb}(A) = \arg\max_{I \in \mathbb{F}^I} d(I,A), \tag{3.31}$$

such that

$$Amb(T_{amb}(A)) = Amb(A). \tag{3.32}$$

Thus, assuming $I = [a,b]$, we seek the solution of the minimization problem

$$d([a,b],A) \to \min,$$

where

$$\int_0^1 \alpha(A_U(\alpha) - A_L(\alpha))d\alpha = \frac{b-a}{2}.$$

The above problem is equivalent to find a,b such that

$$\left(\int_0^1 (A_L(\alpha) - a)^2 d\alpha + \int_0^1 (A_U(\alpha) - b)^2 d\alpha\right) \to \min, \tag{3.33}$$

$$\frac{b-a}{2} = \int_0^1 \alpha(A_U(\alpha) - A_L(\alpha))d\alpha \tag{3.34}$$

$$a \le b. \tag{3.35}$$

After simple calculations our problem (3.33)-(3.35) reduces to

$$\left(2a^2 + \left(4\int_0^1 \alpha A_U(\alpha)d\alpha - 4\int_0^1 \alpha A_L(\alpha)d\alpha \right.\right.$$
$$\left.\left. -2\int_0^1 A_U(\alpha)d\alpha - 2\int_0^1 A_L(\alpha)d\alpha\right) a\right) \to \min,$$
$$b = 2\int_0^1 \alpha(A_U(\alpha) - A_L(\alpha))d\alpha + a$$

and the following result is immediate

$$a = \int_0^1 \left(\alpha + \frac{1}{2}\right) A_L(\alpha)d\alpha + \int_0^1 \left(-\alpha + \frac{1}{2}\right) A_U(\alpha)d\alpha \tag{3.36}$$

$$b = \int_0^1 \left(-\alpha + \frac{1}{2}\right) A_L(\alpha)d\alpha + \int_0^1 \left(\alpha + \frac{1}{2}\right) A_U(\alpha)d\alpha. \tag{3.37}$$

By (1.79) and (1.72) we obtain a nice result:

Theorem 3.5. *([32]) The nearest interval to $A \in \mathbb{F}(\mathbb{R})$, which preserves the ambiguity of A is given by $T_{amb}(A) = [a,b]$ such that*

$$T_{amb}(A) = [EV(A) - Amb(A), EV(A) + Amb(A)]. \tag{3.38}$$

Example 3.1. Consider two fuzzy numbers: $A = (1,2,3,4)$ and $B = (1,2,3,4)_{2,2}$. Hence $A_\alpha = [1+\alpha, 4-\alpha]$ and $B_\alpha = [1+\sqrt{\alpha}, 4-\sqrt{\alpha}]$. Using different interval approximation operators discussed in last two sections we get the following intervals: $T_G(A) = \left[\frac{3}{2}, \frac{7}{2}\right]$, $T_G(B) = \left[\frac{5}{3}, \frac{10}{3}\right]$, $T_{amb}(A) = \left[\frac{5}{3}, \frac{10}{3}\right]$ and $T_{amb}(B) = \left[\frac{9}{5}, \frac{16}{5}\right]$. □

3.4 Triangular approximations of fuzzy numbers

Triangular approximation of fuzzy numbers is usually treated as a special case of the trapezoidal approximation. Here we also do not devote too much time on this type of approximation restricting ourselves to two particular situations connected with the approximation under some specific restrictions.

Firstly let us mention an interesting paper by Ma et al. [149] where the triangular approximation is considered as a step of the defuzzification. The basic idea of the method suggested in [149] is to obtain a symmetric triangular fuzzy number nearest to the fuzzy quantity under study. Although so defined problem exceeds our interests (indeed, generally a fuzzy quantity might not be a fuzzy number), we describe it here in a way suitable to our context.

Let us consider a fuzzy number A. Our goal is to find a symmetric triangular fuzzy number $T(A)$ such that $T(A) = (t_1, t_2, t_3)$, where $t_3 - t_2 = t_2 - t_1 > 0$, which minimizes the Euclidean distance between A and $T(A)$. Thus we seek the solution of the minimization problem

$$d((t_1,t_2,t_3),A) \to \min,$$

where

$$t_3 - t_2 = t_2 - t_1 > 0,$$

which is equivalent to finding $x_0 := t_2$ and $\delta := t_3 - t_2 = t_2 - t_1 > 0$ such that

$$D(x_0,\delta) = \left(\int_0^1 (A_L(\alpha) - x_0 - \sigma + \sigma\alpha)^2 d\alpha \right. \qquad (3.39)$$
$$\left. + \int_0^1 (A_U(\alpha) - x_0 + \sigma - \sigma\alpha)^2 d\alpha \right) \to \min .$$

After simple algebra we get the following solution:

$$x_0 = \frac{1}{2}\int_0^1 (A_L(\alpha) + A_U(\alpha))\, d\alpha \qquad (3.40)$$

$$\sigma = \frac{3}{2}\int_0^1 (A_U(\alpha) - A_L(\alpha))\,(1-\alpha) d\alpha \qquad (3.41)$$

Looking on the problem from the perspective of defuzzification the suggested approach delivers not only a desired real value (i.e. x_0) but also provides an index to determine the fuzziness of the original quantity (i.e. σ).

By (1.72) (1.79) and (1.82) we obtain the following result:

Theorem 3.6. *A symmetric triangular fuzzy number nearest to $A \in \mathbb{F}(\mathbb{R})$ with respect to the Euclidean distance is given by $T(A) = (t_1,t_2,t_3)$ such that*

$$t_1 = EV(A) - \frac{3}{2}(w(A) - Amb(A)) \qquad (3.42)$$

$$t_2 = EV(A) \qquad (3.43)$$

$$t_3 = EV(A) + \frac{3}{2}(w(A) - Amb(A)). \qquad (3.44)$$

An approximation of fuzzy numbers by triangular fuzzy numbers preserving ambiguity was considered by Ban and Coroianu [32]. They proved, in particular, the following theorem.

Theorem 3.7. *([32]) The nearest (with respect to the Euclidean distance) symmetric triangular fuzzy number preserving ambiguity of the fuzzy number A is given by $T(A) = (t_1,t_2,t_3)$ such that*

$$t_1 = EV(A) - 3Amb(A) \qquad (3.45)$$

$$t_2 = EV(A) \qquad (3.46)$$

$$t_3 = EV(A) + 3Amb(A). \qquad (3.47)$$

For the details, we refer the reader to [22, 32].

3.5 Trapezoidal approximations of fuzzy numbers

3.5.1 Trapezoidal approximations preserving expected interval

This section is devoted to trapezoidal approximation of fuzzy numbers, which undoubtedly plays the central role among various approaches to fuzzy number approximation. It is so because, as it was mentioned in Section 3.1, trapezoidal fuzzy numbers form a reasonable compromise between complexity and simplicity in different aspects including data processing and management, calculations, applications and interpretation of fuzzy notions.

Suppose we want to approximate a fuzzy number by a trapezoidal fuzzy number. Thus we need an operator $T : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^T(\mathbb{R})$ which transforms a family of all fuzzy numbers into a family of trapezoidal fuzzy numbers, i.e. $T : A \longmapsto T(A)$.

It is clear that one can approximate a fuzzy number by a trapezoidal one in infinite number of ways. For example, the simplest idea is to substitute a fuzzy number A by $T(A) = (t_1, t_2, t_3, t_4)$ designed by the borders of the support and core, i.e. $t_1 = \inf \operatorname{supp}(A)$, $t_2 = \inf \operatorname{core}(A)$, $t_3 = \sup \operatorname{core}(A)$ and $t_4 = \sup \operatorname{supp}(A)$. However, as it was motivated in Section 3.2, a suitable approximation operator should possess some desired properties and fulfill some necessary and minimal requirements. Delgado et al. [81] in their paper also claim that the approximation should preserve at least some parameters of the original fuzzy number. It was shown in many paper that the approximation operator that guarantees many desired properties can be obtained as the operator T which produces a trapezoidal fuzzy number $T(A)$ that is closest with respect to given distance to the original fuzzy number A among all trapezoidal fuzzy numbers having identical expected interval as the original one. It is so since the invariance of the expected interval often implies many other advantages.

Thus our goal is to find the operator $T : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^T(\mathbb{R})$ which produces for any $A \in \mathbb{F}(\mathbb{R})$ a trapezoidal fuzzy number $T(A)$ nearest to A with respect to the popular Euclidean metric (1.40), i.e.

$$T(A) = \arg \max_{B \in \mathbb{F}^T(\mathbb{R})} d(B, A) \tag{3.48}$$

such that

$$EI(T(A)) = EI(A). \tag{3.49}$$

The above mentioned problem was suggested firstly in [115, 117], developed in [118] and considered also in [119]. Its proper final solution containing four possible operators $T_i(A) = T_i(t_1, t_2, t_3, t_4)$, $i = 1, \ldots, 4$, was proposed independently in [17, 108, 195]. Let us formulate this solution as the following theorem.

Theorem 3.8. *The nearest trapezoidal approximation operator preserving expected interval is such operator $T : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^T(\mathbb{R})$ which for any fuzzy number A assigns the trapezoidal fuzzy number $T(A) = T(t_1, t_2, t_3, t_4)$ as follows*

(a) *if $Amb(A) \geq \frac{1}{3}w(A)$ then*

$$t_1 = -6\int_0^1 \alpha A_L(\alpha)\, d\alpha + 4\int_0^1 A_L(\alpha)\, d\alpha, \tag{3.50}$$

$$t_2 = 6\int_0^1 \alpha A_L(\alpha)\, d\alpha - 2\int_0^1 A_L(\alpha)\, d\alpha, \tag{3.51}$$

$$t_3 = 6\int_0^1 \alpha A_U(\alpha)\, d\alpha - 2\int_0^1 A_U(\alpha)\, d\alpha, \tag{3.52}$$

$$t_4 = -6\int_0^1 \alpha A_U(\alpha)\, d\alpha + 4\int_0^1 A_U(\alpha)\, d\alpha; \tag{3.53}$$

(b) *if $Amb(A) < \frac{1}{3}w(A)$ and $EV_{\frac{1}{3}}(A) \leq Val(A) \leq EV_{\frac{2}{3}}(A)$ then*

$$t_1 = 3\int_0^1 A_L(\alpha)d\alpha - 3\int_0^1 \alpha A_L(\alpha)\, d\alpha - 3\int_0^1 \alpha A_U(\alpha)\, d\alpha + \int_0^1 A_U(\alpha)d\alpha, \tag{3.54}$$

$$t_2 = t_3 = 3\int_0^1 \alpha A_L(\alpha)\, d\alpha + 3\int_0^1 \alpha A_U(\alpha)\, d\alpha - \int_0^1 A_L(\alpha)d\alpha - \int_0^1 A_U(\alpha)d\alpha, \tag{3.55}$$

$$t_4 = 3\int_0^1 A_U(\alpha)d\alpha - 3\int_0^1 \alpha A_L(\alpha)\, d\alpha - 3\int_0^1 \alpha A_U(\alpha)\, d\alpha + \int_0^1 A_L(\alpha)d\alpha; \tag{3.56}$$

(c) *if $Amb(A) < \frac{1}{3}w(A)$ and $Val(A) < EV_{\frac{1}{3}}(A)$ then*

$$t_1 = t_2 = t_3 = \int_0^1 A_L(\alpha)d\alpha, \tag{3.57}$$

$$t_4 = 2\int_0^1 A_U(\alpha)d\alpha - \int_0^1 A_L(\alpha)d\alpha; \tag{3.58}$$

(d) *if $Amb(A) < \frac{1}{3}w(A)$ and $Val(A) > EV_{\frac{2}{3}}(A)$ then*

$$t_1 = 2\int_0^1 A_L(\alpha)d\alpha - \int_0^1 A_U(\alpha)d\alpha, \tag{3.59}$$

$$t_2 = t_3 = t_4 = \int_0^1 A_U(\alpha)d\alpha. \tag{3.60}$$

Proof. As $T(A) = (t_1, t_2, t_3, t_4)$ is a trapezoidal fuzzy number, its α-cuts have a following form $[t_1 + (t_2 - t_1)\alpha, t_4 - (t_4 - t_3)\alpha]$, where $\alpha \in (0, 1]$. Therefore, we have to minimize

$$d(A,T(A)) =$$
$$= \sqrt{\int_0^1 [A_L(\alpha) - (t_1 + (t_2 - t_1)\,\alpha)]^2\, d\alpha + \int_0^1 [A_U(\alpha) - (t_4 - (t_4 - t_3)\alpha)]^2\, d\alpha},$$

with respect to t_1, t_2, t_3, t_4. Moreover, we are looking for such operator satisfying the expected invariance condition (3.49) which, by (1.71), we can rewrite as follows

$$\left[\frac{t_1 + t_2}{2}, \frac{t_3 + t_4}{2}\right] = \left[\int_0^1 A_L(\alpha)d\alpha, \int_0^1 A_U(\alpha)d\alpha\right].$$

Finally, we have to assure that the output would be indeed a trapezoidal fuzzy number, i.e. $t_1 \le t_2 \le t_3 \le t_4$. Last requirement might be expressed by the following inequalities

$$t_1 - t_2 \le 0,$$
$$t_2 - t_3 \le 0,$$
$$t_3 - t_4 \le 0.$$

It is easily seen that it suffices to minimize a function $f(\mathbf{t}) = f(t_1, t_2, t_3, t_4) = d^2(A, T(A))$. Thus, to sum up, we wish to minimize function

$$f(\mathbf{t}) = \int_0^1 [A_L(\alpha) - (t_1 + (t_2 - t_1)\,\alpha)]^2\, d\alpha + \int_0^1 [A_U(\alpha) - (t_4 - (t_4 - t_3)\alpha)]^2\, d\alpha$$

subject to

$$\mathbf{h}(\mathbf{t}) = \left[t_1 + t_2 - 2\int_0^1 A_L(\alpha)d\alpha, t_3 + t_4 - 2\int_0^1 A_U(\alpha)d\alpha\right]^T = \mathbf{0}^T,$$
$$\mathbf{g}(\mathbf{t}) = [t_1 - t_2, t_2 - t_3, t_3 - t_4] \le 0,$$

where $\mathbf{t} \in \mathbb{R}^4$.

By the Karush-Kuhn-Tucker theorem, if $\mathbf{t}^*$ is a local minimizer for the problem of minimizing f subject to $\mathbf{h}(\mathbf{t}) = \mathbf{0}$ and $\mathbf{g}(\mathbf{t}) \le 0$, then there exist the Lagrange multiplier vector λ and the Karush-Kuhn-Tucker multiplier ξ such that

$$\nabla f(\mathbf{t}^*) + \lambda^T \nabla \mathbf{h}(\mathbf{t}^*) + \xi^T \nabla \mathbf{g}(\mathbf{t}^*) = \mathbf{0}^T, \tag{3.61}$$
$$\xi^T \mathbf{g}(\mathbf{t}^*) = 0, \tag{3.62}$$
$$\xi \ge 0. \tag{3.63}$$

In our case, after some calculations, we get

$$\nabla f(\mathbf{t}^*) = \Big[\frac{2}{3}t_1 + \frac{1}{3}t_2 + 2\int_0^1 \alpha A_L(\alpha)\, d\alpha - 2\int_0^1 A_L(\alpha)\, d\alpha,$$
$$\frac{1}{3}t_1 + \frac{2}{3}t_2 - 2\int_0^1 \alpha A_L(\alpha)\, d\alpha, \frac{2}{3}t_3 + \frac{1}{3}t_4 - 2\int_0^1 \alpha A_U(\alpha)\, d\alpha,$$
$$\frac{1}{3}t_3 + \frac{2}{3}t_4 + 2\int_0^1 \alpha A_U(\alpha)\, d\alpha, \frac{2}{3}t_3 - 2\int_0^1 \alpha A_U(\alpha)\, d\alpha\Big],$$

$$\nabla \mathbf{h}(\mathbf{t}^*) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$\nabla \mathbf{g}(\mathbf{t}^*) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Therefore, we can rewrite the Karush-Kuhn-Tucker conditions in a following way

$$\frac{2}{3}t_1 + \frac{1}{3}t_2 + 2\int_0^1 \alpha A_L(\alpha)\, d\alpha - 2\int_0^1 A_L(\alpha)\, d\alpha + \lambda_1 + \xi_1 = 0, \tag{3.64}$$

$$\frac{1}{3}t_1 + \frac{2}{3}t_2 - 2\int_0^1 \alpha A_L(\alpha)\, d\alpha + \lambda_1 - \xi_1 + \xi_2 = 0, \tag{3.65}$$

$$\frac{2}{3}t_3 + \frac{1}{3}t_4 - 2\int_0^1 \alpha A_U(\alpha)\, d\alpha + \lambda_2 - \xi_2 + \xi_3 = 0, \tag{3.66}$$

$$\frac{1}{3}t_3 + \frac{2}{3}t_4 + 2\int_0^1 \alpha A_U(\alpha)\, d\alpha - 2\int_0^1 A_U(\alpha)\, d\alpha + \lambda_2 - \xi_3 = 0, \tag{3.67}$$

$$t_1 + t_2 - 2\int_0^1 A_L(\alpha) d\alpha = 0, \tag{3.68}$$

$$t_3 + t_4 - 2\int_0^1 A_U(\alpha) d\alpha = 0, \tag{3.69}$$

$$\xi_1(t_1 - t_2) = 0, \tag{3.70}$$

$$\xi_2(t_2 - t_3) = 0, \tag{3.71}$$

$$\xi_3(t_3 - t_4) = 0, \tag{3.72}$$

$$\xi_1 \geq 0, \tag{3.73}$$

$$\xi_2 \geq 0, \tag{3.74}$$

$$\xi_3 \geq 0. \tag{3.75}$$

To find points that satisfy the above conditions, we first try $\xi_1 = \xi_2 = \xi_3 = 0$. Then the system of equations (3.64)-(3.75) reduces to following six linear equations

$$\frac{2}{3}t_1 + \frac{1}{3}t_2 + 2\int_0^1 \alpha A_L(\alpha)\,d\alpha - 2\int_0^1 A_L(\alpha)\,d\alpha + \lambda_1 = 0,$$
$$\frac{1}{3}t_1 + \frac{2}{3}t_2 - 2\int_0^1 \alpha A_L(\alpha)\,d\alpha + \lambda_1 = 0,$$
$$\frac{2}{3}t_3 + \frac{1}{3}t_4 - 2\int_0^1 \alpha A_U(\alpha)\,d\alpha + \lambda_2 = 0,$$
$$\frac{1}{3}t_3 + \frac{2}{3}t_4 + 2\int_0^1 \alpha A_U(\alpha)\,d\alpha - 2\int_0^1 A_U(\alpha)\,d\alpha + \lambda_2 = 0,$$
$$t_1 + t_2 - 2\int_0^1 A_L(\alpha)d\alpha = 0,$$
$$t_3 + t_4 - 2\int_0^1 A_U(\alpha)d\alpha = 0.$$

Solving the above equations we obtain

$$t_1 = -6\int_0^1 \alpha A_L(\alpha)\,d\alpha + 4\int_0^1 A_L(\alpha)\,d\alpha, \tag{3.76}$$
$$t_2 = 6\int_0^1 \alpha A_L(\alpha)\,d\alpha - 2\int_0^1 A_L(\alpha)\,d\alpha, \tag{3.77}$$
$$t_3 = 6\int_0^1 \alpha A_U(\alpha)\,d\alpha - 2\int_0^1 A_U(\alpha)\,d\alpha, \tag{3.78}$$
$$t_4 = -6\int_0^1 \alpha A_U(\alpha)\,d\alpha + 4\int_0^1 A_U(\alpha)\,d\alpha, \tag{3.79}$$
$$\lambda_1 = 0,$$
$$\lambda_2 = 0.$$

Now suppose $\xi_2 = \xi_3 = 0$ and $\xi_1 > 0$, which by (3.70) and (3.68) implies

$$t_1 = t_2 = \int_0^1 A_L(\alpha)d\alpha. \tag{3.80}$$

Substituting (3.80) into (3.64) and (3.65) we get

$$\lambda_1 = 0,$$
$$\xi_1 = \int_0^1 A_L(\alpha)d\alpha - 2\int_0^1 \alpha A_L(\alpha)\,d\alpha.$$

However, it is not difficult to see that inequality $\int_0^1 A_L(\alpha)d\alpha - 2\int_0^1 \alpha A_L(\alpha)\,d\alpha > 0$ does not hold in general which contradicts the assumption that $\xi_1 > 0$. Hence, there is no solution for $\xi_2 = \xi_3 = 0$ and $\xi_1 > 0$. In a similar way one may conclude that assuming $\xi_2 = 0$ the solution exists if and only if both $\xi_1 = 0$ and $\xi_3 = 0$.

Now let us suppose that $\xi_2 > 0$. Thus by (3.71) we get immediately $t_2 = t_3$. Assume firstly that $\xi_1 = \xi_3 = 0$. The system of equations (3.64)-(3.75) reduces to following six linear equations:

$$\frac{2}{3}t_1+\frac{1}{3}t_2+2\int_0^1 \alpha A_L(\alpha)\,d\alpha-2\int_0^1 A_L(\alpha)\,d\alpha+\lambda_1=0,$$
$$\frac{1}{3}t_1+\frac{2}{3}t_2-2\int_0^1 \alpha A_L(\alpha)\,d\alpha+\lambda_1+\xi_2=0,$$
$$\frac{2}{3}t_2+\frac{1}{3}t_4-2\int_0^1 \alpha A_U(\alpha)\,d\alpha+\lambda_2-\xi_2=0,$$
$$\frac{1}{3}t_2+\frac{2}{3}t_4+2\int_0^1 \alpha A_U(\alpha)\,d\alpha-2\int_0^1 A_U(\alpha)\,d\alpha+\lambda_2=0,$$
$$t_1+t_2-2\int_0^1 A_L(\alpha)d\alpha=0,$$
$$t_2+t_4-2\int_0^1 A_U(\alpha)d\alpha=0.$$

Solving the above system of equations we get

$$t_1=3\int_0^1 A_L(\alpha)d\alpha-3\int_0^1 \alpha A_L(\alpha)\,d\alpha-3\int_0^1 \alpha A_U(\alpha)\,d\alpha+\int_0^1 A_U(\alpha)d\alpha, \tag{3.81}$$

$$t_2=t_3=3\int_0^1 \alpha A_L(\alpha)\,d\alpha+3\int_0^1 \alpha A_U(\alpha)\,d\alpha-\int_0^1 A_L(\alpha)d\alpha-\int_0^1 A_U(\alpha)d\alpha, \tag{3.82}$$

$$t_4=3\int_0^1 A_U(\alpha)d\alpha-3\int_0^1 \alpha A_L(\alpha)\,d\alpha-3\int_0^1 \alpha A_U(\alpha)\,d\alpha+\int_0^1 A_L(\alpha)d\alpha, \tag{3.83}$$

$$\lambda_1=\frac{1}{3}\int_0^1 A_L(\alpha)d\alpha-\int_0^1 \alpha A_L(\alpha)\,d\alpha+\int_0^1 \alpha A_U(\alpha)\,d\alpha-\frac{1}{3}\int_0^1 A_U(\alpha)d\alpha,$$
$$\lambda_2=-\frac{1}{3}\int_0^1 A_L(\alpha)d\alpha+\int_0^1 \alpha A_L(\alpha)\,d\alpha-\int_0^1 \alpha A_U(\alpha)\,d\alpha+\frac{1}{3}\int_0^1 A_U(\alpha)d\alpha,$$
$$\xi_2=2\int_0^1 \alpha A_L(\alpha)\,d\alpha-2\int_0^1 \alpha A_U(\alpha)\,d\alpha-\frac{2}{3}\int_0^1 A_L(\alpha)d\alpha+\frac{2}{3}\int_0^1 A_U(\alpha)d\alpha. \tag{3.84}$$

However, by the assumption that $\xi_2>0$ and (3.84), we have a legitimate solution to the Karush-Kuhn-Tucker conditions if and only if

$$\frac{1}{3}\left(\int_0^1 A_U(\alpha)d\alpha-\int_0^1 A_L(\alpha)d\alpha\right)>\int_0^1 \alpha A_U(\alpha)d\alpha-\int_0^1 \alpha A_L(\alpha)d\alpha \tag{3.85}$$

and then we get a solution $\mathbf{t}=(t_1,t_2,t_3,t_4)$, given by (3.81)-(3.83).

Now let us consider a situation when not only $\xi_2>0$ but also $\xi_1>0$ and still $\xi_3=0$. Then by (3.68)-(3.71) we get immediately

$$t_1=t_2=t_3=\int_0^1 A_L(\alpha)d\alpha \tag{3.86}$$

and

$$t_4 = 2\int_0^1 A_U(\alpha)d\alpha - \int_0^1 A_L(\alpha)d\alpha. \quad (3.87)$$

Thus we get another solution $\mathbf{t} = (t_1, t_2, t_3, t_4)$, given by (3.86)-(3.87), provided $\xi_2 > 0$, i.e. if inequality (3.85) holds, and

$$\xi_1 = \frac{2}{3}\int_0^1 A_L(\alpha)d\alpha + \frac{1}{3}\int_0^1 A_L(\alpha)d\alpha - \int_0^1 \alpha(A_L(\alpha) + A_U(\alpha))d\alpha > 0. \quad (3.88)$$

We may also consider another situation when $\xi_2 > 0$, $\xi_3 > 0$ and $\xi_1 = 0$ which leads to the fourth solution

$$t_1 = 2\int_0^1 A_L(\alpha)d\alpha - \int_0^1 A_U(\alpha)d\alpha, \quad (3.89)$$

$$t_2 = t_3 = t_4 = \int_0^1 A_U(\alpha)d\alpha, \quad (3.90)$$

which holds provided inequalities (3.85) and

$$\xi_3 = \int_0^1 \alpha(A_L(\alpha) + A_U(\alpha))d\alpha - \frac{1}{3}\int_0^1 A_L(\alpha)d\alpha - \frac{2}{3}\int_0^1 A_U(\alpha)d\alpha > 0 \quad (3.91)$$

are fulfilled.

Finally one may ask what happen if $\xi_1 > 0$, $\xi_2 > 0$ and still $\xi_3 > 0$. But it is seen immediately that this situation has no sense.

Now we have to verify that all our solutions $\mathbf{t} = (t_1, t_2, t_3, t_4)$ satisfy the second-order sufficient conditions. For this we form a matrix

$$L(\mathbf{t}, \lambda, \xi) = \nabla^2 f(\mathbf{t}) + [\lambda \nabla^2 \mathbf{h}(\mathbf{t})] + \xi \nabla^2 g(\mathbf{t}), \quad (3.92)$$

where $[\lambda \nabla^2 \mathbf{h}(\mathbf{t})] = \lambda_1 \nabla^2 h_1(\mathbf{t}) + \lambda_2 \nabla^2 h_2(\mathbf{t})$ and $\nabla^2 h_i(\mathbf{t})$ is the Hessian of $h_i(\mathbf{t})$. One check easily that for our four solutions $\mathbf{t}$ we have $\mathbf{y}^T L(\mathbf{t}, \lambda, \xi)\mathbf{y} > 0$ for all vectors $\mathbf{y}$ in the tangent space to the surface defined by active constraints, i.e. $\{\mathbf{y} : \nabla \mathbf{h}(\mathbf{t})\mathbf{y} = \mathbf{0}, \nabla^2 g(\mathbf{t})\mathbf{y} = 0\}$.

Therefore, we conclude that we have received four different solutions which lead to the nearest trapezoidal fuzzy number that preserves the expected value of the original fuzzy number. Theses solutions are the outputs of four different trapezoidal approximation operators: $T_i(A) = T_i(t_1, t_2, t_3, t_4)$, $i = 1, 2, 3, 4$, where T_1 denotes the approximation operator given by (3.50)-(3.53) that lead to a trapezoidal (but not triangular) fuzzy number, T_2 stands for the operator given by (3.54)-(3.56) that leads to triangular fuzzy number with two sides, T_3 given by (3.57)-(3.58) produces a triangular fuzzy number with the right side only and T_4 given by (3.59)-(3.60) produces a triangular fuzzy number with the left side only. Which one should be used in a particular situation depends on a given fuzzy number, i.e. it depends on conditions (3.85), (3.88) and (3.91) that seem to be very artificial and technical. To make them more clear and to get a better interpretation of those conditions.

Firstly, let us notice that by (1.79) and (1.82) we can rewrite condition (3.85) as

$$Amb(A) < \frac{1}{3}w(A). \tag{3.93}$$

It corresponds to situations when we approximate A by a triangular fuzzy number. It means that for less vague fuzzy numbers the solution is always a triangular fuzzy number.

Now the proper choice of the operator (T_2, T_3 or T_4) depends also on the location of the typical value of the fuzzy number. In particular, using the notion of the weighted expected value (1.73) for $q = \frac{1}{3}$ and the so-called value of a fuzzy number (1.78), we can rewrite condition (3.88) in a following way

$$Val(A) < EV_{\frac{1}{3}}(A).$$

It might be interpreted in such a way that a fuzzy number with a slight ambiguity and which typical value is located closely to the left border of its support would be approximated by a trapezoidal fuzzy number with the right side only, produced by the operator T_3.

Similarly, substituting the value of a fuzzy number (1.78) and the weighted expected value (1.73) for $q = \frac{2}{3}$ in (3.91) we get

$$Val(A) > EV_{\frac{2}{3}}(A),$$

which means that a fuzzy number with a slight ambiguity and which typical value is located closely to the right border of its support would be approximated by a trapezoidal fuzzy number with the left side only, produced by the operator T_4. All other fuzzy numbers with a slight ambiguity, i.e. satisfying (3.93), but simultaneously neither too skew to the left nor to the right, would be approximated by the operator T_2.

On the other hand, more vague fuzzy numbers A satisfying the inverse of (3.93), i.e.

$$Amb(A) \geq \frac{1}{3}w(A),$$

will be approximated by the operator T_1 leading to trapezoidal but not triangular fuzzy numbers. This ends the proof. ■

To sum up we get a following algorithm for computing the nearest trapezoidal approximation preserving the expected interval.

Algorithm 1 *For any $A \in \mathbb{F}(\mathbb{R})$*

Step 1. *If $Amb(A) \geq \frac{1}{3}w(A)$ then apply operator T_1 given by (3.50)-(3.53), else*

Step 2. *if $EV_{\frac{1}{3}}(A) \leq Val(A) \leq EV_{\frac{2}{3}}(A)$ then apply operator T_2 given by (3.54)-(3.56), else*

Step 3. *if $Val(A) < EV_{\frac{1}{3}}(A)$ then apply operator T_3 given by (3.57)-(3.58), else*

Step 4. *apply operator T_4 given by (3.59)-(3.60).*

Let us now consider some examples illustrating different possible situations described in Theorem 3.8.

Example 3.2. Suppose a fuzzy number A has the following membership function

$$\mu_A(x) = \begin{cases} 1-x^2 & \text{if} \quad -1 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

It can be shown that $Amb(A) = \frac{8}{15} > \frac{1}{3}w(A) = \frac{4}{9}$. Thus we apply the trapezoidal approximation operator $T_1(A)$ given by (3.50)–(3.53) and we get a trapezoidal fuzzy number $T_1(A) = (-\frac{16}{15}, -\frac{4}{15}, \frac{4}{15}, \frac{16}{15})$. Membership functions of A and its trapezoidal approximation $T_1(A)$ are given in Figure 3.1.

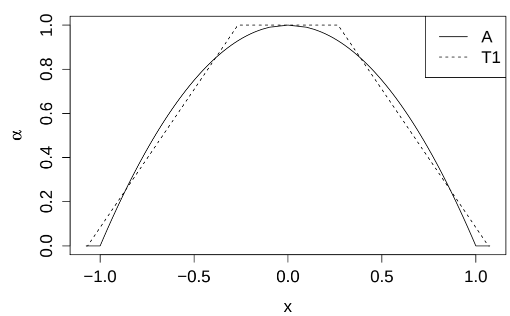

Fig. 3.1 Membership functions of A and its trapezoidal approximation $T_1(A)$, see Example 3.2.

Example 3.3. Let us consider a fuzzy number A with membership function

$$\mu_A(x) = \begin{cases} (x+1)^2 & \text{if} \quad -1 \le x \le 0, \\ (1-x)^2 & \text{if} \quad 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

One can easily see that $Amb(A) = \frac{1}{5} < \frac{1}{3}w(A) = \frac{2}{9}$. Moreover, since A is symmetrical around zero we get immediately $Val(A) = 0$, while $EV_{\frac{2}{3}}(A) = \frac{1}{9}$ and $EV_{\frac{1}{3}}(A) = -\frac{1}{9}$. Theorem 3.8, (b) is applicable, so the nearest trapezoidal approximation is given by (3.54)-(3.56). Finally we obtain a trapezoidal fuzzy number $T_2(A) = (-\frac{2}{3}, 0, 0, \frac{2}{3})$. Membership functions of A and its trapezoidal approximation $T_3(A)$ are given in Figure 3.2. □

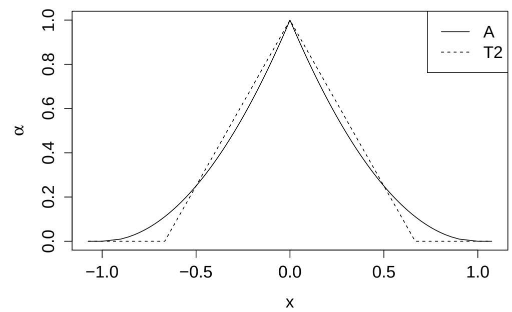

Fig. 3.2 Membership functions of A and its trapezoidal approximation $T_3(A)$, see Example 3.3.

Example 3.4. Now let us consider a fuzzy number A with membership function

$$\mu_A(x) = \begin{cases} (x+1)^2 & \text{if} \quad -1 \le x \le 0, \\ 1 & \text{if} \quad 0 \le x \le 1, \\ (\frac{40-x}{39})^2 & \text{if} \quad 1 \le x \le 40, \\ 0 & \text{otherwise.} \end{cases}$$

We get $Amb(A) = \frac{9}{2} < \frac{1}{3}w(A) = \frac{43}{9}$ and $Val(A) = \frac{43}{10} < EV_{\frac{1}{3}}(A) = \frac{40}{9}$ which means that case (c) in Theorem 3.8 is applicable. Thus the nearest trapezoidal approximation is given by (3.57)-(3.58). Finally we obtain a triangular fuzzy number $T_3(A) = (-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{85}{3})$.Membership functions of A and its trapezoidal approximation $T_3(A)$ are given in Figure 3.3. □

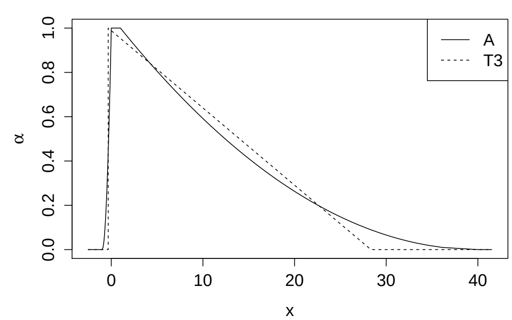

Fig. 3.3 Membership functions of A and its trapezoidal approximation $T_3(A)$, see Example 3.4.

Example 3.5. Consider also another fuzzy number A with membership function

$$\mu_A(x) = \begin{cases} (\frac{x+40}{39})^2 & \text{if} \quad -40 \le x \le -1, \\ 1 & \text{if} \quad -1 \le x \le 0, \\ (1-x)^2 & \text{if} \quad 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Here we get as before $Amb(A) = \frac{9}{2} < \frac{1}{3}w(A) = \frac{43}{9}$ but now $Val(A) = -\frac{43}{10} > EV_{\frac{2}{3}}(A) = -\frac{40}{9}$. Since conditions in Theorem 3.8, (d) hold, the nearest trapezoidal approximation is given by (3.59)-(3.60) and we obtain a triangular fuzzy number $T_4(A) = (-\frac{85}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Membership functions of A and its trapezoidal approximation $T_4(A)$ are given in Figure 3.4. □

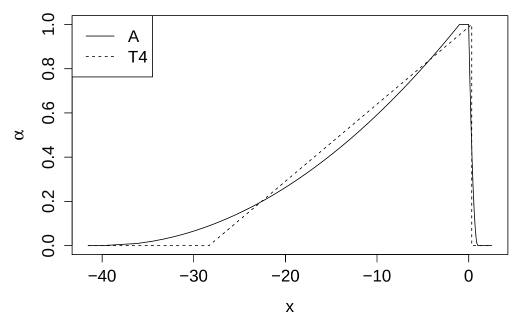

Fig. 3.4 Membership functions of A and its trapezoidal approximation $T_4(A)$, see Example 3.5.

We can also obtain an equivalent algorithm for choosing a proper approximation operator using parameter $\bar{y}(A)$ called the y-coordinate of the centroid point of a fuzzy number A. In [185] the authors showed that

$$\bar{y}(A) = \frac{\int_0^1 \alpha(A_U(\alpha) - A_L(\alpha))d\alpha}{\int_0^1 (A_U(\alpha) - A_L(\alpha))d\alpha}. \tag{3.94}$$

It is easily see that

$$\bar{y}(A) = \frac{Amb(A)}{w(A)}. \tag{3.95}$$

Therefore, we get immediately that our condition (a) in Theorem 3.8 is equivalent to the following one

$$\bar{y}(A) \ge \frac{1}{3}. \tag{3.96}$$

It means that we approximate a fuzzy number A by the trapezoidal approximation operator T_1 if the y-coordinate of the centroid point of A is not smaller than one third. Otherwise, we apply operator T_2 or T_3 or T_4. The consecutive steps for choosing a suitable operator remains as before. Thus we get another algorithm (which is in fact a conjunction of our Algorithm 1 the algorithm given in [196]).

Algorithm 2 *For any* $A \in \mathbb{F}(\mathbb{R})$

Step 1. *If* $\overline{y}(A) \geq \frac{1}{3}$ *then apply operator* T_1 *given by (3.50)-(3.53), else*
Step 2. *if* $EV_{\frac{1}{3}}(A) \leq Val(A) \leq EV_{\frac{2}{3}}(A)$ *then apply operator* T_2 *given by (3.54)-(3.56), else*
Step 3. *if* $Val(A) < EV_{\frac{1}{3}}(A)$ *then apply operator* T_3 *given by (3.57)-(3.58), else*
Step 4. *apply operator* T_4 *given by (3.59)-(3.60).*

Since one can propose many approximation methods for fuzzy numbers the question about the quality of approximation is of importance. However, before we discuss the properties of the method given in Theorem 3.8 let us notice that - according to that theorem - we can consider the family $\mathbb{F}(\mathbb{R})$ of all fuzzy numbers as a union of four subfamilies $\mathbb{F}_i(\mathbb{R})$ corresponding to different approximation operators to be used. Namely, we may say that a fuzzy number A belongs to subfamily $\mathbb{F}_i(\mathbb{R})$ if and only if T_i $(i = 1, \ldots, 4)$ is an appropriate operator that should be used for getting a proper trapezoidal approximation. Thus, by the considerations given above and some simple calculations we get following lemmas (see [110, 111]).

Lemma 3.1. *The following conditions are equivalent:*

(a) $A \in \mathbb{F}_1(\mathbb{R})$,
(b) $Amb(A) \geq \frac{1}{3} w(A)$,
(c) $\overline{y}(A) \geq \frac{1}{3}$.

Lemma 3.2. *The following conditions are equivalent:*

(a) $A \in \mathbb{F}_2(\mathbb{R})$,
(b) $Amb(A) < \frac{1}{3} w(A)$ *and* $EV_{\frac{1}{3}}(A) \leq Val(A) \leq EV_{\frac{2}{3}}(A)$,
(c) $\overline{y}(A) < \frac{1}{3}$ *and* $EV_{\frac{1}{3}}(A) \leq Val(A) \leq EV_{\frac{2}{3}}(A)$,
(d) $Amb(A) < \frac{1}{3} w(A)$ *and* $|EV(A) - Val(A)| \leq \frac{1}{6} w(A)$,
(e) $Amb(A) < \frac{1}{3} w(A)$ *and* $|\Delta Amb(A)| \leq \frac{1}{6} w(A)$,

where $\Delta Amb(A) = Amb_U(A) - Amb_L(A)$, *while the right-hand ambiguity* $Amb_U(A)$ *and the left-hand ambiguity* $Amb_L(A)$ *are given by* (1.81) *and* (1.80), *respectively.*

Lemma 3.3. *The following conditions are equivalent:*

(a) $A \in \mathbb{F}_3(\mathbb{R})$,
(b) $Val(A) < EV_{\frac{1}{3}}(A)$,
(c) $Val(A) < EV(A) + \frac{1}{6}w(A)$,
(d) $\Delta Amb(A) < \frac{1}{6}w(A)$.

Lemma 3.4. *The following conditions are equivalent:*

(a) $A \in \mathbb{F}_4(\mathbb{R})$,
(b) $Val(A) > EV_{\frac{2}{3}}(A)$,
(c) $Val(A) > EV(A) + \frac{1}{6}w(A)$,
(d) $\Delta Amb(A) > \frac{1}{6}w(A)$.

One may notice that subfamilies $\mathbb{F}_1(\mathbb{R}),\ldots,\mathbb{F}_4(\mathbb{R})$ form a partition of a family of all fuzzy numbers $\mathbb{F}(\mathbb{R})$. Actually, by lemmas given above, we may conclude immediately that

$$\mathbb{F}_1(\mathbb{R}) \cup \ldots \cup \mathbb{F}_4(\mathbb{R}) = \mathbb{F}(\mathbb{R})$$

and

$$\mathbb{F}_i(\mathbb{R}) \cap \mathbb{F}_j(\mathbb{R}) = \emptyset \quad \text{for} \quad i \neq j.$$

Introducing this useful notation we can now turn back to properties of the trapezoidal approximation operators. It can be shown that for $A \in \mathbb{F}_i(\mathbb{R})$ the nearest trapezoidal approximation operator T_i, $i = 1,\ldots,4$, preserving the expected interval is invariant to translations and scale invariant, is monotonic and fulfills identity criterion, preserves the expected interval and fulfills the nearness criterion with respect to metric (1.40) in subfamily of all trapezoidal fuzzy numbers with fixed expected interval. Moreover, it is continuous and compatible with the extension principle, is order invariant with respect to some preference fuzzy relations, is correlation invariant and it preserves the width. For more details we refer the reader to [20, 118, 119].

Now let us consider the behavior of the value and ambiguity of fuzzy number under trapezoidal approximation operators T_i, $i = 1,\ldots,4$. According to [20, 118, 119] we know that these two parameters are not generally preserved. The following theorem clarifies the situation for all four approximation operators.

Theorem 3.9. *([110]) Let $A \in \mathbb{F}(\mathbb{R})$ and $T_1, T_2, T_3, T_4 : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^T(\mathbb{R})$ denote the nearest trapezoidal approximation operators preserving the expected interval, given in Theorem 3.8. Then:*

(a) *if $A \in \mathbb{F}_1(\mathbb{R})$, then*

$$Val(T_1(A)) = Val(A),$$
$$Amb(T_1(A)) = Amb(A),$$

(b) *if $A \in \mathbb{F}_2(\mathbb{R})$, then*

$$Val(T_2(A)) = Val(A),$$
$$Amb(T_2(A)) > Amb(A),$$

(c) *if* $A \in \mathbb{F}_3(\mathbb{R})$*, then*

$$Val(T_3(A)) > Val(A),$$
$$Amb(T_3(A)) > Amb(A),$$

(d) *if* $A \in \mathbb{F}_4(\mathbb{R})$*, then*

$$Val(T_4(A)) < Val(A),$$
$$Amb(T_4(A)) > Amb(A).$$

Thus the value of a fuzzy number is preserved by T_1 and T_2 only, while the ambiguity is invariant solely under T_1. Directly from the proof of Theorem 3.9 we get a following corollary.

Corollary 3.5. *Let* $A \in \mathbb{F}(\mathbb{R})$*. Then*

(a) $Val(T_3(A)) = EV_{\frac{1}{3}}(A)$ *for* $A \in \mathbb{F}_3(\mathbb{R})$*,*
(b) $Val(T_4(A)) = EV_{\frac{2}{3}}(A)$ *for* $A \in \mathbb{F}_4(\mathbb{R})$*,*
(c) $Amb(T_i(A)) = \frac{1}{3}w(A)$ *for* $A \in \mathbb{F}_i(\mathbb{R})$*,* $i = 2,3,4$*.*

One may be also interested whether the y-coordinate of the centroid point of a fuzzy number A is invariant under the nearest trapezoidal approximation operators preserving the expected interval.

Theorem 3.10. *([110]) Let* $A \in \mathbb{F}(\mathbb{R})$*. Then*

(a) $\overline{y}(T_1(A)) = \overline{y}(A)$ *for* $A \in \mathbb{F}_1(\mathbb{R})$*,*
(b) $\overline{y}(T_i(A)) = \frac{1}{3} > \overline{y}(A)$ *for* $A \in \mathbb{F}_i(\mathbb{R})$*,* $i = 2,3,4$*.*

One may also ask if the weighted expected value EV_q, $q \in [0,1]$, defined by (1.73), is preserved by the nearest trapezoidal approximation operators discussed above. It turn out, that this parameter - contrary to value, ambiguity or the y-coordinate of the centroid point of a fuzzy number - is preserved by all four approximation operators.

Theorem 3.11. *([110]) If* $A \in \mathbb{F}_i(\mathbb{R})$*,* $i = 1,2,3,4$*, then for each* $q \in [0,1]$

$$EV_q(T_i(A)) = EV_q(A). \tag{3.97}$$

For the proofs of the above theorems we refer the reader to [110].

In the end of this section let us also mention some interesting distance properties between fuzzy numbers and their trapezoidal approximations preserving the expected interval. Ban and Coroianu [27] proved that the trapezoidal approximation operator which preserves the expected interval satisfies the Lipschitz property.

Moreover, since in that case, by Theorem 3.8, we actually have four different approximation operators, each one corresponding to a subfamily of fuzzy numbers (see Theorem 3.9 and the discussion above), we may find the best Lipschitz constant for each subfamily $\mathbb{F}_i(\mathbb{R})$, $i = 1,2,3,4$.

Theorem 3.12. *Let A and B denote two fuzzy numbers such that*

$$\int_0^1 B_U(\alpha)d\alpha - \int_0^1 B_L(\alpha)d\alpha \geq \int_0^1 A_U(\alpha)d\alpha - \int_0^1 A_L(\alpha)d\alpha. \tag{3.98}$$

Then:

(a) *if* $A \in \mathbb{F}_1(\mathbb{R})$, *then*

$$d(T(A), T(B)) \leq d(A,B) \tag{3.99}$$

(b) *if* $A \in \mathbb{F}_2(\mathbb{R})$, *then*

$$d(T(A), T(B)) \leq \frac{2\sqrt{3}}{3} d(A,B) \tag{3.100}$$

(c) *if* $A \in \mathbb{F}_3(\mathbb{R})$, *then*

$$d(T(A), T(B)) \leq \sqrt{\frac{5}{3}} d(A,B) \tag{3.101}$$

(d) *if* $A \in \mathbb{F}_4(\mathbb{R})$, *then*

$$d(T(A), T(B)) \leq \sqrt{\frac{5}{3}} d(A,B). \tag{3.102}$$

For the proof we refer the reader to [71], Corollary 9.

It is worth noting that the results given above are not only interesting from the theoretical perspective but might appear useful in practice because they allow to approximate fuzzy numbers with an acceptable error in the case when the direct formula is difficult to apply. Other details on the continuity of the trapezoidal fuzzy number-valued operators can be found in [27, 29].

3.5.2 Trapezoidal approximations with restrictions on support and core

Unfortunately, for very skew membership functions our optimal approximation operators may reveal very unpleasant behavior. Namely, if $Val(A) < EV_{1/3}(A)$ or $Val(A) > EV_{2/3}(A)$, then it may happen that the core of the corresponding operator's output $T_3(A)$ or $T_4(A)$, respectively, is disjoint with the core of the original fuzzy number A. To illustrate such situation let us consider the following example.

Example 3.6. By Example 3.4 we get α-cuts $A_\alpha = [\sqrt{\alpha} - 1, 40 - 39\sqrt{\alpha}]$ of the fuzzy number A and its nearest trapezoidal approximation $T_3(A) = (-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{85}{3})$. It is easily seen that $\text{core}(A) = [0,1]$, $\text{core}(T(A)) = \{-\frac{1}{3}\}$ and hence $\text{core}(A) \cap \text{core}(T(A)) = \emptyset$ (see also Figure 3.3). □

It is obvious that the solution shown in Example 3.6 although mathematically proper is completely unacceptable from the practical point of view. Actually, disjoint cores of the input and output means that the approximation indicates other "sure" element than the original fuzzy number. To avoid such undesired situations Abbasbandy and Hajjari [5] considered trapezoidal approximation preserving the core of a fuzzy number, i.e. satisfying

$$\text{core}(T(A)) = \text{core}(A). \tag{3.103}$$

Although this idea sometimes might be useful, some objections remain. Actually, $\mu_A(x) = 1$ leads to perfect information that x surely belongs to A. If $\mu_A(x)$ is close to 1 we say that x rather belongs to A. And conversely, $\mu_A(x) = 0$ shows that x surely does not belong to A (and belongs to $\neg A$) which is also a perfect information. Similarly, x such that $\mu_A(x)$ is close to 0 is interpreted as a point that rather does not belong to A. However, if $\mu_A(x)$ is close to 0.5 we dot know how to classify x because it belong to A and to its completion $\neg A$ more or less with the same degree. Thus, degrees of membership both high (close to 1) and low (close to 0) are much more informative than those close to 0.5 and hence there is no reason to favor the core only and to discriminate against the support. So one may also argue the need of the support preservation, i.e.

$$\text{supp}(T(A)) = \text{supp}(A). \tag{3.104}$$

Unfortunately, if we fix both the core and support we obtain a naive approximation $T(A) = T(a_1, a_2, a_3, a_4)$ which neglects completely the shape of the sides of the original fuzzy number A.

Therefore, to evade this trap we have to weaken slightly requirements (3.103) and (3.104). However, we should do it without loosing the main idea at the same time. It seems that the right direction was suggested in [103], Def. 2, or [112], i.e. instead of the core and support preservation the appropriate inclusions would be desirable. Namely, it seems that the sufficiently strong requirement to be satisfied is to substitute equalities in (3.103) and (3.104) by the appropriate inclusions, i.e.

$$\text{core}(A) \subseteq \text{core}(T(A)) \tag{3.105}$$

$$\text{supp}(T(A)) \subseteq \text{supp}(A) \tag{3.106}$$

or, using notation adopted above, we may demand for

$$[a_2, a_3] \subseteq [t_2, t_3] \tag{3.107}$$

$$[t_1, t_4] \subseteq [a_1, a_4]. \tag{3.108}$$

These conditions reflect our concern that each point which surely belongs to A would also belongs to its approximation $T(A)$ and that each point which surely does not belong to A would also not belong to $T(A)$.

Now, having in mind the above discussion we will try to improve the solution delivered in Section 3.5.1 so that conditions (3.105) and (3.106) for the core and support would be also fulfilled. Therefore, combing these requirements with the problem (3.48)-(3.49) considered above, our current problem is to find $T^*(A) \in \mathbb{F}^T(\mathbb{R})$ such that

$$d(A, T^*(A)) = \min_{T \in \mathbb{F}^T(\mathbb{R})} d(A, T), \tag{3.109}$$

with respect to the following conditions:

$$EI(T^*(A)) = EI(A), \tag{3.110}$$

$$\text{core}(A) \subseteq \text{core}(T^*(A)) \tag{3.111}$$

$$\text{supp}(T^*(A)) \subseteq \text{supp}(A). \tag{3.112}$$

Since the α-cut of a trapezoidal fuzzy number is equal to $[t_1 + (t_2 - t_1)\alpha, t_4 - (t_4 - t_3)\alpha]$, thus our goal might be expressed as follows: find t_1, t_2, t_3, t_4 which minimize

$$D(t_1, t_2, t_3, t_4) = \int_0^1 [A_L(\alpha) - (t_1 + (t_2 - t_1)\,\alpha)]^2\, d\alpha + \int_0^1 [A_U(\alpha) - (t_4 - (t_4 - t_3)\alpha)]^2\, d\alpha$$

with respect to conditions

$$\left[\frac{t_1 + t_2}{2}, \frac{t_3 + t_4}{2}\right] = \left[\int_0^1 A_L(\alpha) d\alpha, \int_0^1 A_U(\alpha) d\alpha\right],$$

$$a_1 \le t_1 \le t_2 \le a_2 \le a_3 \le t_3 \le t_4 \le a_4.$$

Further on we assume that $a_1 < a_2$ and $a_3 < a_4$ because otherwise the linearization of the left or right side has no sense.

Since $a_2 \le a_3$ are fixed, it is easily seen that the above stated problem can be solved as two separate minimization programs. Denoting by $EI_L(A) = \int_0^1 A_L(\alpha) d\alpha$ and $EI_U(A) = \int_0^1 A_U(\alpha) d\alpha$ the left and right border of the expected interval $EI(A)$, respectively, we may express these two programs as follows:

Program 1: find t_1 and t_2 which minimize

$$f_1(t_1, t_2) = \int_0^1 [A_L(\alpha) - (t_1 + (t_2 - t_1)\,\alpha)]^2\, d\alpha$$

such that

$$\frac{t_1 + t_2}{2} - EI_L(A) = 0,$$

$$a_1 \le t_1 \le t_2 \le a_2.$$

Program 2: find t_3 and t_4 which minimize

$$f_2(t_3,t_4) = \int_0^1 [A_U(\alpha) - (t_4 - (t_4 - t_3)\alpha)]^2 \, d\alpha$$

such that

$$\frac{t_3+t_4}{2} - EI_U(A) = 0,$$
$$a_3 \le t_3 \le t_4 \le a_4.$$

Similarly as in Section 3.5.1 to get a solution of Program 1 we apply the Karush-Kuhn-Tucker theorem for the local minimizer of $f_1(\mathbf{t})$ subject to $h_1(\mathbf{t}) = t_1 + t_2 - 2EI_L(A) = 0$ and $g_1(\mathbf{t}) = [a_1 - t_1, t_1 - t_2, t_2 - a_2] \le 0$. Then there exist the Lagrange multiplier vector λ and the Karush-Kuhn-Tucker multiplier ξ such that

$$\nabla f_1(\mathbf{t}^*) + \lambda^T \nabla \mathbf{h}_1(\mathbf{t}^*) + \xi^T \nabla \mathbf{g}_1(\mathbf{t}^*) = \mathbf{0}^T, \qquad (3.113)$$
$$\xi^T \mathbf{g}_1(\mathbf{t}^*) = 0, \qquad (3.114)$$
$$\xi \ge 0. \qquad (3.115)$$

After some calculations and discussion similarly to those performed in Section 3.5.1, we get the solution expressed by the following proposition.

Proposition 3.1. *Let A denote a fuzzy number with the support* $[a_1, a_4]$ *and core* $[a_2, a_3]$. *Then Program 1 has four possible solutions depending on A, given as follows:*

(a) *if* $a_1 \le 4EI_L(A) - 6\int_0^1 \alpha A_L(\alpha)d\alpha$ *and* $6\int_0^1 \alpha A_L(\alpha)d\alpha - 2EI_L(A) \le a_2$ *then* $t_1 = 4EI_L(A) - 6\int_0^1 \alpha A_L(\alpha)d\alpha$ *and* $t_2 = 6\int_0^1 \alpha A_L(\alpha)d\alpha - 2EI_L(A)$

(b) *if* $4EI_L(A) - 6\int_0^1 \alpha A_L(\alpha)d\alpha < a_1$ *and* $6\int_0^1 \alpha A_L(\alpha)d\alpha - 2EI_L(A) \le a_2$ *then* $t_1 = a_1$ *and* $t_2 = 2EI_L(A) - a_1$

(c) *if* $a_1 \le 4EI_L(A) - 6\int_0^1 \alpha A_L(\alpha)d\alpha$ *and* $a_2 < 6\int_0^1 \alpha A_L(\alpha)d\alpha - 2EI_L(A)$ *then* $t_1 = 2EI_L(A) - a_2$ *and* $t_2 = a_2$

(d) *if* $4EI_L(A) - 6\int_0^1 \alpha A_L(\alpha)d\alpha < a_1$ *and* $a_2 < 6\int_0^1 \alpha A_L(\alpha)d\alpha - 2EI_L(A)$ *then* $t_1 = a_1$ *and* $t_2 = a_2$.

For the proof we refer the reader to [123].

Looking on Proposition 3.1 one immediately conclude that the particular solution depends on the relationship between the lower bounds of the support and core of the fuzzy number under study and two values $4EI_L(A) - 6\int_0^1 \alpha A_L(\alpha)d\alpha$ and $6\int_0^1 \alpha A_L(\alpha)d\alpha - 2EI_L(A)$, respectively. Thus let us try to find a suitable interpretation for these two critical values. It seems that the notion of the left spread $LSP(A)$ of a fuzzy number A might be helpful there (see [122]). We define $LSP(A)$ by

$$LSP(A) = 6\int_0^1 (\alpha - \frac{1}{2}) A_L(\alpha)\, d\alpha \qquad (3.116)$$
$$= 6\int_0^1 \alpha A_L(\alpha)d\alpha - 3EI_L(A).$$

It can be shown (see [123]) that $LSP(A) \geq 0$ for any fuzzy number A. To get some intuition related to (3.116) please note, that if A is a rectangular fuzzy number, then $LSP(A) = 0$, which is obvious since A corresponds to the crisp interval and hence its membership function does not change gradually. If A is a trapezoidal fuzzy number, i.e. $A = A(a_1, a_2, a_3, a_4)$, then $LSP(A) = \frac{1}{2}(a_2 - a_1)$, which is equal to the half of the range of the left side of the membership function μ_A describing A. Hence, longer the distance between a_1 and a_2, where μ_A is increasing, the greater is the left spread $LSP(A)$. It is clear, that the right side of a fuzzy number might be characterized in a similar way, by the upper spread introduced later in (3.122).

It is easily seen that using this notion we can express the conditions given in Proposition 3.1 in a more friendly way, i.e.

$$4EI_L(A) - 6\int_0^1 \alpha A_L(\alpha)d\alpha = EI_L(A) - LSP(A), \tag{3.117}$$

$$6\int_0^1 \alpha A_L(\alpha)d\alpha - 2EI_L(A) = EI_L(A) + LSP(A). \tag{3.118}$$

In a similar way we can solve Program 2 devoted to the right side of a fuzzy number. We apply again the Karush-Kuhn-Tucker theorem for the local minimizer of $f_2(\mathbf{t})$ subject to $h_2(\mathbf{t}) = t_3 + t_4 - 2EI_U(A) = 0$ and $g_2(\mathbf{t}) = [a_3 - t_3, t_3 - t_4, t_4 - a_4] \leq 0$. Then there exist the Lagrange multiplier vector η and the Karush-Kuhn-Tucker multiplier τ such that

$$\nabla f_2(\mathbf{t}^*) + \eta^T \nabla \mathbf{h}_2(\mathbf{t}^*) + \tau^T \nabla \mathbf{g}_2(\mathbf{t}^*) = \mathbf{0}^T, \tag{3.119}$$

$$\tau^T \mathbf{g}_2(\mathbf{t}^*) = 0, \tag{3.120}$$

$$\tau \geq 0. \tag{3.121}$$

Now, as in the case of Program 1, we have to consider different situations and finally we get again four possible solutions of Program 2 depending of the particular shape of the original fuzzy number A. Here we omit these tedious calculations. Moreover, to express the solutions in a nice form we utilize the upper spread $USP(A)$ of a fuzzy number A defined by (see [122])

$$\begin{aligned} USP(A) &= 6\int_0^1 (\frac{1}{2} - \alpha)A_U(\alpha)\, d\alpha \\ &= 3EI_U(A) - 6\int_0^1 \alpha A_U(\alpha)d\alpha. \end{aligned} \tag{3.122}$$

It can be shown that $USP(A) \geq 0$ for any fuzzy number A.

Finally, summing up the above discussion we may conclude that the following theorem describes the desired approximation of a fuzzy number.

Theorem 3.13. *For any fuzzy number A there exist a unique trapezoidal approximation $T^*(A) = (t_1, t_2, t_3, t_4)$ of A closest to A with respect to metric* (1.40) *and preserving the expected interval with restrictions on the support and core, i.e. satisfying* *conditions* (3.109)-(3.112). *Moreover, if* $\mathrm{supp}(A) = [a_1, a_4]$ *and* $\mathrm{core}(A) = [a_2, a_3]$, *then the left side of the trapezoidal approximation, i.e.* t_1 *and* t_2*, are given by*

(1) *if* $a_1 \leq EI_L(A) - LSP(A)$ *and* $EI_L(A) + LSP(A) \leq a_2$ *then*

$$t_1 = EI_L(A) - LSP(A) \tag{3.123}$$
$$t_2 = EI_L(A) + LSP(A) \tag{3.124}$$

(2) *if* $EI_L(A) - LSP(A) < a_1$ *and* $EI_L(A) + LSP(A) \leq a_2$ *then*

$$t_1 = a_1 \tag{3.125}$$
$$t_2 = 2EI_L(A) - a_1 \tag{3.126}$$

(3) *if* $a_1 \leq EI_L(A) - LSP(A)$ *and* $a_2 < EI_L(A) + LSP(A)$ *then*

$$t_1 = 2EI_L(A) - a_2 \tag{3.127}$$
$$t_2 = a_2 \tag{3.128}$$

(4) *if* $EI_L(A) - LSP(A) < a_1$ *and* $a_2 < EI_L(A) + LSP(A)$ *then*

$$t_1 = a_1 \tag{3.129}$$
$$t_2 = a_2, \tag{3.130}$$

while the right side, i.e. t_3 *and* t_4*, are given as by*

(5) *if* $a_3 \leq EI_U(A) - USP(A)$ *and* $EI_U(A) + USP(A) \leq a_4$ *then*

$$t_3 = EI_U(A) - USP(A) \tag{3.131}$$
$$t_4 = EI_U(A) + USP(A) \tag{3.132}$$

(6) *if* $EI_U(A) - USP(A) < a_3$ *and* $EI_L(A) + USP(A) \leq a_4$ *then*

$$t_3 = a_3 \tag{3.133}$$
$$t_4 = 2EI_U(A) - a_3 \tag{3.134}$$

(7) *if* $a_3 \leq EI_U(A) - USP(A)$ *and* $a_4 < EI_L(A) + USP(A)$ *then*

$$t_3 = 2EI_U(A) - a_4 \tag{3.135}$$
$$t_4 = a_4 \tag{3.136}$$

(8) *if* $EI_U(A) - USP(A) < a_3$ *and* $a_4 < EI_L(A) + USP(A)$ *then*

$$t_3 = a_3 \tag{3.137}$$
$$t_4 = a_4. \tag{3.138}$$

Before discussing properties of the trapezoidal approximation preserving the expected interval with restrictions on the support and core, suggested in the previous section, let us go back to Examples 3.4 and 3.6 which gave the motivations for our considerations.

Example 3.7. One can easily compute basic characteristics of the fuzzy number considered in Example 3.4 and required for choosing the proper trapezoidal approximation. Namely, $EI_L(A) = -\frac{1}{3}$, $EI_U(A) = 14$, $LSP(A) = \frac{2}{5}$ and $USP(A) = \frac{78}{5}$. Since $EI_L(A) - LSP(A) = -\frac{11}{15} > a_1 = -1$ and $EI_L(A) + LSP(A) = \frac{1}{15} > a_2 = 0$, hence by Theorem 3.13 $t_1 = 2EI_L(A) - a_2 = -\frac{2}{3}$ and $t_2 = a_2 = 0$. Moreover, since $EI_U(A) - USP(A) = -\frac{8}{5} < a_3 = 1$ and $EI_U(A) + USP(A) = \frac{148}{5} < a_4 = 40$, we get $t_3 = a_3 = 1$ and $t_4 = 2EI_U(A) - a_3 = 27$.

Therefore, finally the desired trapezoidal approximation of our fuzzy number A is $T(A) = (-\frac{2}{3}, 0, 1, 27)$. It is easily seen that $\text{core}(A) = \text{core}(T(A))$ and $\text{supp}(T(A)) \subset \text{supp}(A)$. Membership functions of A and its trapezoidal approximation $T(A)$ are given in Figure 3.5. □

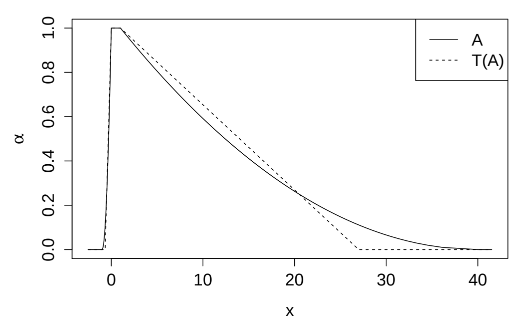

Fig. 3.5 Membership functions of A and its trapezoidal approximation $T(A)$, see Example 3.7.

One may ask about the relationship between trapezoidal approximation described in this and in the previous section, i.e. with and without restrictions on support and core. Some light in this topic is delivered by the following lemma (see [123]).

Lemma 3.5. *Suppose A is a fuzzy number with* $\text{supp}(A) = [a_1, a_4]$ *and* $\text{core}(A) = [a_2, a_3]$ *and such that*

$$
\begin{aligned}
a_1 &\leq EI_L(A) - LSP(A) \\
a_2 &\geq EI_L(A) + LSP(A) \\
a_3 &\leq EI_U(A) - USP(A) \\
a_4 &\geq EI_U(A) + USP(A).
\end{aligned}
$$

Then the trapezoidal approximation $T^*(A)$ of A closest to A with respect to metric (1.40) *and preserving the expected interval with restrictions on the support and core is equal to the trapezoidal approximation operator T_1 given by* (3.50)-(3.53) *in Theorem 3.8 without requirements on the support and core, i.e. $T^*(A) = T_1(A)$.*

The proof is evident. However, the interpretation of Lemma 3.5 is worth commenting. Namely, operator T_1 is chosen by Theorem 3.8 (or, equivalently by Algorithms 1 or 2) if and only if the input fuzzy number A is not too asymmetrical. These four conditions mentioned in Lemma 3.5 also characterize a fuzzy number with a slight or moderate asymmetry. Thus it is clear that additional restrictions on the support and core (3.105)-(3.106) are crucial only for fuzzy numbers with skew membership functions.

Our trapezoidal approximation operator T^* has many desired properties. Some of them are listed below.

Lemma 3.6. *([123], Lemma 4) Let T^* denote the trapezoidal approximation closest to the input fuzzy number with respect to metric* (1.40) *and preserving the expected interval with restrictions on the support and core, given by Theorem 3.13. Then*

1) T^* is invariant to translations, i.e. $T^*(A+z) = T^*(A) + z$ $(\forall z \in \mathbb{R})$;
2) T^* fulfills the identity criterion, i.e. $T^*(A) = A$ $(\forall A \in \mathbb{F}^T(\mathbb{R}))$;
3) T^* preserves the expected value, i.e. $EV(T^*(A)) = EV(A)$;
4) T^* preserves the width, i.e. $w(T^*(A)) = w(A)$.

We omit the proof. It is straightforward but requires tedious calculations for each combination of possible left and right sides of a fuzzy number we may get by the Theorem 3.13. Last two properties are immediate because they are just the consequence of the expected interval invariance. Some other desired properties of the trapezoidal approximation operators are described and discussed e.g. in [118, 119].

Although the expected interval invariance in the trapezoidal approximation is hardly recommended because it guaranties many useful properties, in some situation the user would like to omit this requirement. However, even then we should look not only for the trapezoidal fuzzy number nearest to the original one but have to keep restrictions (3.105)-(3.106) on the support and core. Otherwise it may happen that the output would be inadmissible, similarly as in Example 3.6.

Therefore, we should solve the following problem: given a fuzzy number A find a trapezoidal approximation $T^{**}(A)$ such that

$$d(A, T^{**}(A)) = \min_{T \in \mathbb{F}^T(\mathbb{R})} d(A, T), \tag{3.139}$$

$$\operatorname{core}(A) \subseteq \operatorname{core}(T^{**}(A)) \tag{3.140}$$

$$\operatorname{supp}(T^{**}(A)) \subseteq \operatorname{supp}(A). \tag{3.141}$$

It is clear that our problem now is quite similar to that discussed above. So we will not go into details but only sketch the reasoning and present final solution.

As before our problem might be solved as two separate minimization programs:

Program 3: find t_1 and t_2 which minimize

$$f_1(t_1,t_2) = \int_0^1 [A_L(\alpha) - (t_1 + (t_2 - t_1)\,\alpha)]^2\, d\alpha$$

such that $a_1 \leq t_1 \leq t_2 \leq a_2$.

Program 4: find t_3 and t_4 which minimize

$$f_2(t_3,t_4) = \int_0^1 [A_U(\alpha) - (t_4 - (t_4 - t_3)\alpha)]^2\, d\alpha$$

such that $a_3 \leq t_3 \leq t_4 \leq a_4$.

To solve Program 3 we apply again the Karush-Kuhn-Tucker theorem for the local minimizer of f_1 subject to g_1 and solve the system of equations nearly identical as (3.113)-(3.115) but without function h_1. Similarly, for solving Program 4 we have to consider the system of equations like (3.119)-(3.121) but without function h_2. And finally, we may present the solution of these two programs in the form of the following theorem.

Theorem 3.14. *For any fuzzy number A there exist a unique trapezoidal approximation $T^{**}(A) = (t_1,t_2,t_3,t_4)$ of A closest to A with respect to metric* (1.40) *and preserving the restrictions on the support and core, i.e. satisfying conditions* (3.139)-(3.141). *Moreover, if* $\mathrm{supp}(A) = [a_1,a_4]$ *and* $\mathrm{core}(A) = [a_2,a_3]$*, then the left side of the trapezoidal approximation, i.e. t_1 and t_2, are given by*

(1) If $a_1 \leq EI_L(A) - LSP(A)$ and $EI_L(A) + LSP(A) \leq a_2$ then

$$t_1 = EI_L(A) - LSP(A) \tag{3.142}$$
$$t_2 = EI_L(A) + LSP(A) \tag{3.143}$$

(2) if $EI_L(A) - LSP(A) < a_1$ and $EI_L(A) + LSP(A) \leq a_2$ then

$$t_1 = a_1 \tag{3.144}$$
$$t_2 = \frac{3}{2}EI_L(A) + \frac{1}{2}LSP(A) - \frac{1}{2}a_1 \tag{3.145}$$

(3) if $a_1 \leq EI_L(A) - LSP(A)$ and $a_2 < EI_L(A) + LSP(A)$ then

$$t_1 = \frac{3}{2}EI_L(A) - \frac{1}{2}LSP(A) - \frac{1}{2}a_2 \tag{3.146}$$
$$t_2 = a_2 \tag{3.147}$$

(4) if $EI_L(A) - LSP(A) < a_1$ and $a_2 < EI_L(A) + LSP(A)$ then

$$t_1 = a_1 \tag{3.148}$$
$$t_2 = a_2, \tag{3.149}$$

while the right side, i.e. t_3 and t_4, are given as

(5) *If* $a_3 \le EI_U(A) - USP(A)$ *and* $EI_U(A) + USP(A) \le a_4$ *then*

$$t_3 = EI_U(A) - USP(A) \tag{3.150}$$
$$t_4 = EI_U(A) + USP(A) \tag{3.151}$$

(6) *if* $EI_U(A) - USP(A) < a_3$ *and* $EI_L(A) + USP(A) \le a_4$ *then*

$$t_3 = a_3 \tag{3.152}$$
$$t_4 = \frac{3}{2}EI_U(A) + \frac{1}{2}USP(A) - \frac{1}{2}a_3 \tag{3.153}$$

(7) *if* $a_3 \le EI_U(A) - USP(A)$ *and* $a_4 < EI_L(A) + USP(A)$ *then*

$$t_3 = \frac{3}{2}EI_U(A) - \frac{1}{2}USP(A) - \frac{1}{2}a_4 \tag{3.154}$$
$$t_4 = a_4 \tag{3.155}$$

(8) *if* $EI_U(A) - USP(A) < a_3$ *and* $a_4 < EI_L(A) + USP(A)$ *then*

$$t_3 = a_3 \tag{3.156}$$
$$t_4 = a_4. \tag{3.157}$$

We omit the long proof because it requires calculations very similar to those shown in Section 3.5.1.

Comparing Theorem 3.13 and Theorem 3.14 one easily conclude that in some cases we may obtain the same trapezoidal approximation no matter whether we assume the expected interval invariance or not. Before we formulate a theorem describing the relationship between these two kinds of approximations we have to introduce some notation.

Definition 3.13. Let A denote a fuzzy number with the support $[a_1, a_4]$ and core $[a_2, a_3]$. We sat that

(a) $A \in \mathbb{F}(l_1, \cdot)$ iff $a_1 \le EI_L(A) - LSP(A)$ and $a_2 \ge EI_L(A) + LSP(A)$,
(b) $A \in \mathbb{F}(l_2, \cdot)$ iff $a_1 > EI_L(A) - LSP(A)$ and $a_2 \ge EI_L(A) + LSP(A)$,
(c) $A \in \mathbb{F}(l_3, \cdot)$ iff $a_1 \le EI_L(A) - LSP(A)$ and $a_2 < EI_L(A) + LSP(A)$,
(d) $A \in \mathbb{F}(l_4, \cdot)$ iff $a_1 > EI_L(A) - LSP(A)$ and $a_2 < EI_L(A) + LSP(A))$,
(e) $A \in \mathbb{F}(\cdot, r_1)$ iff $a_3 \le EI_U(A) - USP(A)$ and $a_4 \ge EI_U(A) + USP(A)$,
(f) $A \in \mathbb{F}(\cdot, r_2)$ iff $a_3 > EI_U(A) - USP(A)$ and $a_4 \ge EI_U(A) + USP(A)$,
(g) $A \in \mathbb{F}(\cdot, r_3)$ iff $a_3 \le EI_U(A) - USP(A)$ and $a_4 < EI_U(A) + USP(A)$,
(h) $A \in \mathbb{F}(\cdot, r_4)$ iff $a_3 > EI_U(A) - USP(A)$ and $a_4 < EI_U(A) + USP(A))$.

One can find easily that the subfamilies $\mathbb{F}(l_i, r_j) \subset \mathbb{F}(\mathbb{R})$ for $i = 1, \ldots, 4$ and $j = 1, \ldots, 4$ introduced in Definition 3.13 are disjoint, i.e. $\mathbb{F}(l_i, r_j) \cap \mathbb{F}(l_m, r_n) = \emptyset$ for $i \ne m$ and $j \ne n$. Moreover, these sixteen subfamilies cover the whole family of fuzzy numbers. Indeed, $\bigcup_{i=1}^{4} \bigcup_{j=1}^{4} \mathbb{F}(l_i, r_j) = \mathbb{F}(\mathbb{R})$.

Looking carefully one realize that $\mathbb{F}(l_1,\cdot),\ldots,\mathbb{F}(l_4,\cdot)$ correspond to those subclasses of fuzzy numbers which appear in points (1)-(4) in Theorem 3.13 and Theorem 3.14, while $\mathbb{F}(\cdot,r_1),\ldots,\mathbb{F}(\cdot,r_4)$ to the subclasses discussed in points (5)-(8) of those theorems, respectively. Now we are able to formulate a theorem on the relations (equivalence or inclusion) between trapezoidal fuzzy approximations $T^*(A)$ and $T^{**}(A)$.

Theorem 3.15. *([123], Theorem 7) Suppose A is a fuzzy number with the support* $[a_1,a_4]$ *and core* $[a_2,a_3]$*. Then*

(s1) $T^*(A)=T^{**}(A)$ *if and only if* $A\in\mathbb{F}(l_1,r_1)\cup\mathbb{F}(l_1,r_4)\cup\mathbb{F}(l_4,r_1)\cup\mathbb{F}(l_4,r_4)$.
(s2) $T^*(A)\subset T^{**}(A)$ *if and only if* $A\in\mathbb{F}(l_1,r_2)\cup\mathbb{F}(l_3,r_1)\cup\mathbb{F}(l_3,r_2)\cup\mathbb{F}(l_3,r_4)\cup\mathbb{F}(l_4,r_2)$.
(s3) $T^*(A)\supset T^{**}(A)$ *if and only if* $A\in\mathbb{F}(l_1,r_3)\cup\mathbb{F}(l_2,r_1)\cup\mathbb{F}(l_2,r_3)\cup\mathbb{F}(l_2,r_4)\cup\mathbb{F}(l_4,r_3)$.

Proof. Combining situations described in points (1), (4), (5) and (8) of Theorem 3.13 and Theorem 3.14 we get immediately the equivalence of the approximations, i.e. point (s1) holds.

Now we have to consider fuzzy numbers that appear in points (2), (3), (6) and (7) of Theorem 3.13 and Theorem 3.14. Further on we will adopt the following notation: $T^*(A)=(t_1^*,t_2^*,t_3^*,t_4^*)$ and $T^{**}(A)=(t_1^{**},t_2^{**},t_3^{**},t_4^{**})$.

If $A\in\mathbb{F}(l_2,\cdot)$, i.e. we focus our attention on point (2) in Theorems 3.13 and 3.14 then $t_1^*=a_1=t_1^{**}$ and we get

$$t_2^{**}-t_2^*=\frac{1}{2}a_1-\frac{1}{2}(EI_L(A)-LSP(A))>0,$$

because $a_1>EI_L(A)-LSP(A)$ by Definition 3.13 (b). Hence

$$A\in\mathbb{F}(l_2,\cdot)\Leftrightarrow t_2^*<t_2^{**}. \tag{3.158}$$

Suppose now $A\in\mathbb{F}(l_3,\cdot)$, which corresponds to point (3) in Theorems 3.13 and 3.14. Then $t_2^*=a_2=t_2^{**}$ and we obtain

$$t_1^*-t_1^{**}=\frac{1}{2}(EI_L(A)+LSP(A))-\frac{1}{2}a_2>0,$$

since $a_2<EI_L(A)+LSP(A)$ by Definition 3.13 (c). Thus

$$A\in\mathbb{F}(l_3,\cdot)\Leftrightarrow t_1^*>t_1^{**}. \tag{3.159}$$

If $A\in\mathbb{F}(\cdot,r_2)$, like in point (6) in Theorems 3.13 and 3.14, then $t_3^*=a_3=t_3^{**}$ and

$$t_4^{**}-t_4^*=\frac{1}{2}a_3-\frac{1}{2}(EI_U(A)-USP(A))>0,$$

since by Definition 3.13 (f) we have $a_3>EI_U(A)-USP(A)$. Therefore

$$A \in \mathbb{F}(\cdot, r_2) \Leftrightarrow t_4^* < t_4^{**}. \tag{3.160}$$

Now assume that $A \in \mathbb{F}(\cdot, r_3)$, like in point (7) in Theorems 3.13 and 3.14. Then $t_4^* = a_4 = t_4^{**}$. Moreover,

$$t_3^* - t_3^{**} = \frac{1}{2}(EI_U(A) + USP(A)) - \frac{1}{2}a_4 > 0,$$

because by Definition 3.13 (g) we have $a_4 < EI_U(A) + USP(A)$. So

$$A \in \mathbb{F}(\cdot, r_3) \Leftrightarrow t_3^* > t_3^{**}. \tag{3.161}$$

Since any trapezoidal fuzzy number has linear sides thus the inclusion $T^*(A) \subseteq T^{**}(A)$ holds if and only if $t_1^* \geq t_1^{**}$, $t_2^* \geq t_2^{**}$, $t_3^* \leq t_3^{**}$ and $t_4^* \leq t_4^{**}$. To get strict inclusion, i.e. $T^*(A) \subset T^{**}(A)$, at least one of those four inequalities should be strict. Therefore, summing up cases (3.158)-(3.161), we can conclude that $T^*(A) \subset T^{**}(A)$ holds if and only if the input fuzzy number A belongs to $\mathbb{F}(l_1, r_2) \cup \mathbb{F}(l_3, r_1) \cup \mathbb{F}(l_3, r_2) \cup \mathbb{F}(l_3, r_4) \cup \mathbb{F}(l_4, r_2)$, which proves (s2). The similar reasoning applies for the opposite inclusion (s3), which proves the theorem. ■

Example 3.8. Let us consider once more the fuzzy number A that appears in Examples 3.4 and 3.7. In Example 3.7 we have obtained $T^*(A) = (-\frac{2}{3}, 0, 1, 27)$. Now, by Theorem 3.14 we get $T^{**}(A) = (-0.7, 0, 1, 28.3)$. Here $T^*(A) \subset T^{**}(A)$ because $A \in \mathbb{F}(l_3, r_2)$. □

Please, notice, that Theorem 3.15 describes the relationship between trapezoidal approximations with and without additional requirement on the expected interval invariance for fuzzy numbers belonging to 14 of 16 subfamilies $\mathbb{F}(l_i, r_j)$. Actually, two subfamilies: $\mathbb{F}(l_2, r_2)$ and $\mathbb{F}(l_3, r_3)$ were left. Thus one may wonder what happen for fuzzy numbers from these very two subfamilies. Before stating next theorem let us recall the most natural partial ordering in $\mathbb{F}(\mathbb{R})$ defined as

$$A \prec B \Leftrightarrow (A_L(\alpha) \leq B_L(\alpha), A_U(\alpha) \leq B_U(\alpha), \quad \forall \alpha \in [0,1]), \tag{3.162}$$

i.e. $A \prec B$ when both sides of A are not greater than the sides of B (see [137]).

Theorem 3.16. *([123], Theorem 8) Suppose A is a fuzzy number with the support $[a_1, a_4]$ and core $[a_2, a_3]$. Then*

(i) $T^*(A) \prec T^{**}(A)$ *if and only if* $A \in \mathbb{F}(l_2, r_2)$.
(ii) $T^{**}(A) \prec T^*(A)$ *if and only if* $A \in \mathbb{F}(l_3, r_3)$.

Proof. Suppose $A \in \mathbb{F}(l_2, r_2)$. By Definition 3.13 and Theorems 3.13 and 3.14 we obtain $t_1^* = t_1^{**} = a_1$, $t_2^* < t_2^{**}$, $t_3^* = t_3^{**} = a_3$ and $t_4^* < t_4^{**}$. Hence, because of the linearity of the sides of $T^*(A)$ and $T^{**}(A)$ and (3.162) we have $T^*(A) \prec T^{**}(A)$.

On the other hand, if $A \in \mathbb{F}(l_3, r_3)$ then we get $t_1^* > t_1^{**} = a_1$, $t_2^* = t_2^{**} = a_2$, $t_3^* > t_3^{**}$ and $t_4^* = t_4^{**} = a_4$. This time the linearity of the sides and (3.162) gives $T^{**}(A) \prec T^*(A)$, which completes the proof. ■

Example 3.9. Now suppose a fuzzy number A has the following membership function

$$\mu_A(x) = \begin{cases} (\frac{x-1}{27})^2 & \text{if} \quad 1 \le x \le 28, \\ 1 & \text{if} \quad 28 \le x \le 30, \\ 1 - (\frac{x-30}{20})^2 & \text{if} \quad 30 \le x \le 50, \\ 0 & \text{otherwise.} \end{cases}$$

So we get α-cuts $A_\alpha = [1 + 27\sqrt{\alpha}, 30 + 20\sqrt{1-\alpha}]$ and hence we may compute $EI_L(A) = 19$, $EI_U(A) = \frac{130}{3}$, $LSP(A) = \frac{54}{5}$ and $USP(A) = 8$. We get $EI_L(A) - LSP(A) = \frac{41}{5} > a_1 = 1$, $EI_L(A) + LSP(A) = \frac{149}{5} > a_2 = 28$, $EI_U(A) - USP(A) = \frac{106}{3} > a_3 = 30$ and $EI_U(A) + USP(A) = \frac{154}{3} > a_4 = 50$. Hence $A \in \mathbb{F}(l_3, r_3)$. By Theorems 3.13 and 3.14 we get $T^*(A) = (10, 28, \frac{110}{3}, 50)$ and $T^{**}(A) = (\frac{91}{10}, 28, 36, 50)$. Here $T^{**}(A) \prec T^*(A)$ which is obvious since $A \in \mathbb{F}(l_3, r_3)$. □

3.5.3 Bi-symmetrically weighted trapezoidal approximations of fuzzy numbers

In some situations other distances than metric (1.40) might be more suitable. Using metric (1.40) all α-cuts are treated evenly. This feature is sometimes criticized by authors who claim that elements belonging to α_1-cut should be treated with the higher attention that those from α_2-cut if $\alpha_1 > \alpha_2$ because the membership degree for the first group is higher and so they are less uncertain. This point of view can be found, e.g., in [204], where the trapezoidal approximation with respect to the weighted distance

$$d_{ZL}(A,B) = \left(\int_0^1 \alpha [A_L(\alpha) - B_L(\alpha)]^2 d\alpha + \int_0^1 \alpha [A_U(\alpha) - B_U(\alpha)]^2 d\alpha \right)^{1/2} \tag{3.163}$$

with increasing weighting function is applied (one may notice that (3.163) is a particular version of the weighted L_2-type distance (1.42)).

Although such increasing weighting might be useful in some occasions, another weighted distances would be more interesting in general. This is a straightforward conclusion from the fact that the least informative α-cut is not zero but 0.5. Actually, situation $\mu_A(x) = 1$ leads to perfect information that x surely belongs to A. If $\mu_A(x)$ is close to 1 we'll say that x rather belongs to A. And conversely, $\mu_A(x) = 0$ shows that x surely does not belong to A (and belongs to $\neg A$) which is also a perfect information. Similarly, x such that $\mu_A(x)$ is close to 0 is interpreted as a point that rather does not belong to A. However, if $\mu_A(x) = 0.5$ we dot know how to classify x because it belong to A and to it's completion $\neg A$ with the same degree. The same happens if $\mu_A(x)$ is close to 0.5. Thus, to sum up, degrees of membership both high (close to 1) and low (close to 0) are much more informative than those close to 0.5. Hence, if we try to incorporate this obvious conclusion into practice we have to consider not increasing weighted distance (3.163) but so-called bi-symmetrical weighted distance suggested in [121]. More precisely, the authors of [121] considered the following two bi-symmetrical weighted functions

$$\lambda_1(\alpha) = \begin{cases} 1-2\alpha & \text{if} \quad \alpha \in [0, \frac{1}{2}], \\ 2\alpha - 1 & \text{if} \quad \alpha \in [\frac{1}{2}, 1], \end{cases} \tag{3.164}$$

and

$$\lambda_2(\alpha) = \begin{cases} 1 & \text{if} \quad \alpha \in [0, \frac{1}{4}] \cup [\frac{3}{4}, 1], \\ 0 & \text{if} \quad \alpha \in (\frac{1}{4}, \frac{3}{4}). \end{cases} \tag{3.165}$$

Function (3.164) is in some sense a bi-symmetrical counterpart of the increasing weighted function such as applied in [204]. The second one, contrary to previous continuous weighted function is a noncontinuous one which appreciates only elements with high or low degree of membership and does not take into account the other. In some sense (3.165) corresponds to Pedrycz's viewpoint expressed in his shadowed sets (see [163]) where we consider only these points which rather belong to a set under study or those that rather do not belong to it. The other elements with intermediate membership degree form the so-called shadow.

One can, of course, propose many other weighted functions having properties similar to (3.164) or (3.165). Thus it would be interesting to specify a general definition of the bi-symmetrical weighted function. Then for the defined family of bi-symmetrical weighted functions we will consider trapezoidal approximation based on the bi-symmetrical weighted distance obtained for any representant of that family.

Definition 3.14. Any function $\lambda : [0,1] \to [0,1]$ such that

(a) $\lambda(\frac{1}{2} - \alpha) = \lambda(\frac{1}{2} + \alpha)$ for all $\alpha \in [0, \frac{1}{2}]$,
(b) $\lambda(\frac{1}{2}) \leq \lambda(\alpha)$ for any $\alpha \in [0,1]$,

is called a **bi-symmetrical weighted function**.

Thus each bi-symmetrical weighted function λ is symmetrical around $\frac{1}{2}$ and λ reaches its minimum in $\frac{1}{2}$. Further on we will examine a reach subclass of all bi-symmetrical weighted function, called regular bi-symmetrical weighted functions.

Definition 3.15. A bi-symmetrical weighted function λ is called regular if

(a) $\lambda(\frac{1}{2}) = 0$,
(b) $\lambda(0) = \lambda(1) = 1$,
(c) $\int_0^1 \lambda(\alpha) d\alpha = \frac{1}{2}$.

It is obvious that both (3.164) and (3.165) are regular bi-symmetrical weighted functions. A family of all regular bi-symmetrical weighted functions will be denoted by Λ_{BS}.

Now let us go back to the trapezoidal approximation of fuzzy numbers which produce a trapezoidal fuzzy number $T(A)$ that is the closest to given original fuzzy number A among all trapezoidal fuzzy numbers having identical expected interval as the original one, discussed in Section 3.5.1. However, now we will look for the operators which minimize the weighted distance (1.42) based on bi-symmetrical function, i.e. when $\lambda_L = \lambda_U = \lambda \in \Lambda_{BS}$. Keeping in mind all desired formulae for the expected interval (1.71) and the weighted distance (1.42) and substituting there the α-cut of a trapezoidal fuzzy number $T = (t_1, t_2, t_3, t_4)$ our problem might be expressed as follows:
given $\lambda \in \Lambda_{BS}$ find $t_1, t_2, t_3, t_4 \in \mathbb{R}$ which minimize

$$f(t_1,t_2,t_3,t_4) = \left(\int_0^1 \lambda(\alpha) \left[A_L(\alpha) - (t_1 + (t_2 - t_1)\,\alpha)\right]^2 d\alpha \right. \qquad (3.166)$$

$$\left. + \int_0^1 \lambda(\alpha) \left[A_U(\alpha) - (t_4 - (t_4 - t_3)\alpha)\right]^2 d\alpha \right)^{1/2}$$

with respect to conditions

$$\frac{t_1 + t_2}{2} = \int_0^1 A_L(\alpha) d\alpha, \qquad (3.167)$$

$$\frac{t_3 + t_4}{2} = \int_0^1 A_U(\alpha) d\alpha \qquad (3.168)$$

$$t_1 \le t_2 \le t_3 \le t_4. \qquad (3.169)$$

This problem for bi-symmetrical weighted functions (3.164) and (3.165) was solved in [121], while the general solution for any bi-symmetrical weighted function was given in [122]. To present this general solution in a nice form with a possibly clear interpretation some useful notation should be introduced.

Firstly let us notice that by Definition 3.15 the centroid of a bi-symmetrical weighted function is $\frac{1}{2}$. Therefore, the dispersion of a bi-symmetrical weighted function λ is given by

$$\eta = \int_0^1 (\alpha - \frac{1}{2})^2 \lambda(\alpha)\, d\alpha. \qquad (3.170)$$

We also introduce another two parameters characterizing the dispersion of the left side and of the right side of given fuzzy number A with respect to given bi-symmetrical weighted function λ. Namely, the left (lower) spread of a fuzzy number A with respect to considered bi-symmetrical weighted function λ is a number $LSP_\lambda(A)$ given by

$$LSP_\lambda(A) = \frac{1}{2\eta} \int_0^1 (\alpha - \frac{1}{2}) \lambda(\alpha) A_L(\alpha)\, d\alpha, \qquad (3.171)$$

while the right (upper) spread of a fuzzy number A with respect to considered bi-symmetrical weighted function λ is a following number $USP_{\lambda}(A)$

$$USP_{\lambda}(A) = \frac{1}{2\eta}\int_0^1 (\frac{1}{2}-\alpha)\lambda(\alpha)A_U(\alpha)\,d\alpha. \tag{3.172}$$

It can be shown that $LSP_{\lambda}(A) \geq 0$ and $USP_{\lambda}(A) \geq 0$. Let us also denote the total spread of a given fuzzy number A with respect to considered bi-symmetrical weighted function λ by $TSP_{\lambda}(A)$, i.e.

$$TSP_{\lambda}(A) = LSP_{\lambda}(A) + USP_{\lambda}(A). \tag{3.173}$$

The difference between the right and the left spread of a fuzzy number will be denoted by $\Delta SP_{\lambda}(A)$, i.e.

$$\Delta SP_{\lambda}(A) = USP_{\lambda}(A) - LSP_{\lambda}(A). \tag{3.174}$$

It is easily seen that as $TSP_{\lambda}(A)$ is always nonnegative, while $\Delta SP_{\lambda}(A)$ might be positive or negative as A is more asymmetrical to the right or to the left. Let us also recall that $w(A)$ denotes the width of a fuzzy number A as it was defined by (1.82), while $EI_L(A)$ and $EI_U(A)$ stand for the left and right bound of the expected interval $EI(A)$ of a fuzzy number A, respectively.

We can now formulate our main result in this section, i.e. the solution of problem (3.166)-(3.169).

Theorem 3.17. *([122], Theorem 5)*

For any regular bi-symmetrical weighted function $\lambda \in \Lambda_{BS}$ the nearest trapezoidal approximation operator preserving expected interval with respect to distance (1.42) *based on λ is such operator $T : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^T(\mathbb{R})$, that for any fuzzy number A assigns the trapezoidal fuzzy number $T(A) = (t_1, t_2, t_3, t_4)$, where*

(a) *if $w(A) \geq TSP_{\lambda}(A)$ then*

$$\begin{aligned} t_1 &= EI_L(A) - LSP_{\lambda}(A), \\ t_2 &= EI_L(A) + LSP_{\lambda}(A), \\ t_3 &= EI_U(A) - USP_{\lambda}(A), \\ t_4 &= EI_U(A) + USP_{\lambda}(A); \end{aligned}$$

(b) *if $|\Delta SP_{\lambda}(A)| \leq w(A) < TSP_{\lambda}(A)$ then*

$$\begin{aligned} t_1 &= EI_L(A) - \frac{1}{2}w(A) + \frac{1}{2}\Delta SP_{\lambda}(A), \\ t_2 = t_3 &= EV(A) - \frac{1}{2}\Delta SP_{\lambda}(A), \\ t_4 &= EI_U(A) + \frac{1}{2}w(A) + \frac{1}{2}\Delta SP_{\lambda}(A); \end{aligned}$$

(c) *if* $w(A) < \Delta SP_\lambda(A)$ *then*

$$\begin{aligned} t_1 &= t_2 = t_3 = EI_L(A), \\ t_4 &= 2EI_U(A) - EI_L(A); \end{aligned}$$

(d) *if* $\Delta SP_\lambda(A) < 0$ *and* $w(A) < |\Delta SP_\lambda(A)|$ *then*

$$\begin{aligned} t_1 &= 2EI_L(A) - EI_U(A), \\ t_2 &= t_3 = t_4 = EI_U(A). \end{aligned}$$

To prove this theorem the minimization problem under given constraints should be performed and hence the Karush-Kuhn-Tucker theorem would be useful here, as it was in Section 3.5.1.

It is worth noting that, similarly as in Theorem 3.8, we have received not a single operator but four different operators providing the nearest trapezoidal fuzzy number that preserves the expected value of the original fuzzy number, where T_1, corresponding to point (a), leads to a regular trapezoidal fuzzy number, T_2 corresponding to point (b), stands for the operator that leads to a triangular fuzzy number with two sides, while T_3 and T_4, corresponding to points (c) and (d), produce triangular fuzzy numbers with the right or the left side only, respectively. In other words, we approximate a fuzzy number A by the trapezoidal approximation operator T_1 provided the total dispersion of the given fuzzy number with respect to the considered bi-symmetrical weighted function measured by the sum of the lower and upper spread is large enough. Otherwise, we will approximate A by a triangular fuzzy number. However, for less dispersed fuzzy numbers we have three possible situations: to approximate a fuzzy number A we apply operator T_2 provided the asymmetry of A is not too big (i.e. there is no big difference between the lower and upper spread). If A reveals high right asymmetry (i.e. the right spread is significantly larger than the lower spread) it would be approximated by a triangular fuzzy number with the right side only, produced by operator T_3. Otherwise, a fuzzy number with high left asymmetry would be approximated by a triangular fuzzy number with the left side only, produced by operator T_4.

It is interesting that in the trapezoidal approximation with the bi-symmetrical weighted function we have obtained four possible solutions like in the problem with non-weighted distance considered in Theorem 3.8. Moreover, operators T_3 and T_4 are identical as given in Theorem 3.8. It means that for very asymmetrical fuzzy numbers its nearest trapezoidal approximation preserving the expected interval remains independent whether we use weighted or non-weighted distance.

Looking for parallels with the problem with non-weighted distance considered in Section 3.5.1 we can also notice that the family $\mathbb{F}(\mathbb{R})$ of all fuzzy numbers can be considered as a union of four subfamilies $\mathbb{F}_i(\mathbb{R})$ corresponding to different approximation operators to be used. We may say that a fuzzy number A belongs to subfamily $\mathbb{F}_i(\mathbb{R})$ if and only if T_i $(i = 1, \ldots, 4)$ is an appropriate operator that should be used for getting a proper trapezoidal approximation. One may notice that subfamilies $\mathbb{F}_1(\mathbb{R}),\ldots,\mathbb{F}_4(\mathbb{R})$ form a partition of a family of all fuzzy numbers $\mathbb{F}(\mathbb{R})$, i.e. $\mathbb{F}_1(\mathbb{R})\cup\ldots\cup\mathbb{F}_4(\mathbb{R})=\mathbb{F}(\mathbb{R})$ and $\mathbb{F}_i(\mathbb{R})\cap\mathbb{F}_j(\mathbb{R})=\emptyset$ for $i\neq j$.

3.5.4 Trapezoidal approximations preserving the ambiguity and value

In Section 1.7 we have discussed several characteristics of fuzzy numbers that are often applied to describe in a concise way some specific features of a fuzzy number and to unease their representation and handling. One can find among them the value (1.78) and ambiguity (1.79) of a fuzzy number, that capture the relevant information on its location and dispersion, respectively, introduced in [81].

In the same paper the authors discussed how to approximate a given fuzzy number by a suitable trapezoidal one preserving the value and ambiguity. Since it is not possible to uniquely determine a trapezoidal fuzzy number which is characterized by four numbers by two conditions only, some additional conditions must be introduced. In particular, we may solve the problem by finding the nearest trapezoidal approximation of a fuzzy number with respect to given metric and such that the value and ambiguity are preserved (see [25]).

Thus our goal is to find the operator $T:\mathbb{F}(\mathbb{R})\to\mathbb{F}^T(\mathbb{R})$ which produces for any $A\in\mathbb{F}(\mathbb{R})$ a trapezoidal fuzzy number $T(A)$ nearest to A with respect to the Euclidean metric (1.40), i.e.

$$T(A)=\arg\max_{B\in\mathbb{F}^T(\mathbb{R})} d(B,A) \tag{3.175}$$

such that

$$Val(T(A))=Val(A), \tag{3.176}$$

$$Amb(T(A))=Amb(A). \tag{3.177}$$

It is worth recalling that the nearest trapezoidal approximation operator preserving the expected interval, given by Theorem 3.8, does not preserve in general the value and ambiguity of a fuzzy number (see Theorem 3.9).

Although one can use the Karush-Kuhn-Tucker theorem to solve our problem, here we present another method, maybe less sophisticated but avoiding the laborious calculus. Firstly let us introduce some concepts and notation proposed by Yeh in [195].

An **extended trapezoidal fuzzy number** is an ordered pair of polynomial functions of degree less than or equal to 1. A family of all extended trapezoidal fuzzy numbers is given by

$$\mathbb{F}_e^T(\mathbb{R})=\Big\{T=[l,u,x,y]:T_L(\alpha)=l+x(\alpha-\frac{1}{2}), \tag{3.178}$$

$$T_U(\alpha)=u-y(\alpha-\frac{1}{2}),\ \alpha\in[0,1],\ l,u,x,y\in\mathbb{R}\Big\}$$

One can notice immediately the similarity between the notation used in the above formula and those considered in Section 1.6 for usual fuzzy numbers. Although an extended trapezoidal fuzzy number may not be a fuzzy number, but the Euclidean distance between two extended trapezoidal fuzzy numbers is similarly defined as in (1.40). Moreover, we define the value and the ambiguity of an extended trapezoidal fuzzy number in the same way as in the case of a trapezoidal fuzzy number.

The extended trapezoidal approximation $T_e(A) = [l_e, u_e, x_e, y_e]$ of a fuzzy number A is the extended trapezoidal fuzzy number which minimizes the Euclidean distance $d(A,X)$, where $X \in \mathbb{F}_e^T(\mathbb{R})$ and it is determined by the following formulae:

$$l_e = \int_0^1 A_L(\alpha)d\alpha \tag{3.179}$$

$$u_e = \int_0^1 A_U(\alpha)d\alpha \tag{3.180}$$

$$x_e = 12\int_0^1 (\alpha - \frac{1}{2})A_L(\alpha)d\alpha \tag{3.181}$$

$$y_e = -12\int_0^1 (\alpha - \frac{1}{2})A_U(\alpha)d\alpha \tag{3.182}$$

One may check that x_e and y_e are non-negative real number while $l_e \leq u_e$ by the definition of a fuzzy number.

The following distance properties of the extended trapezoidal approximation operator are of interest (see [30] for a more general approach).

Proposition 3.2. *([194], Proposition 4.2) Let A be a fuzzy number. Then for any trapezoidal fuzzy number B*

$$d^2(A,B) = d^2(A,T_e(A)) + d^2(T_e(A),B).$$

Proposition 3.3. *([194], Proposition 4.4) For any fuzzy numbers A and B*

$$d(T_e(A),T_e(B)) \leq d(A,B).$$

A straightforward calculations show the invariance of the value and ambiguity under the extended trapezoidal approximation. Indeed, the following proposition holds.

Proposition 3.4. *([25], Proposition 4.4) For any fuzzy number A*

$$Val(T_e(A)) = Val(A),$$
$$Amb(T_e(A)) = Amb(A).$$

Let us now return to our main goal of designing the approximation operator $T(A)$ satisfying (3.175)-(3.177). Let us also recall the notation introduced in Section 1.6 and the alternative expression (1.70) of the Euclidean metric (1.40). By Propositions 3.2 and 3.3, $T(A) = [l_T, u_T, x_T, y_T]$ if and only if $(l_T, u_T, x_T, y_T) \in \mathbb{R}^4$ is a solution of the problem

$$\min\left((l-l_e)^2+(u-u_e)^2+\frac{1}{12}(x-x_e)^2+\frac{1}{12}(y-y_e)^2\right) \tag{3.183}$$

under the conditions

$$x \geq 0 \tag{3.184}$$
$$y \geq 0 \tag{3.185}$$
$$x+y \leq 2u-2l \tag{3.186}$$
$$-6l+6u-x-y=-6l_e+6u_e-x_e-y_e \tag{3.187}$$
$$6l+6u+x-y=6l_e+6u_e+x_e-y_e, \tag{3.188}$$

where l_e,u_e,x_e,y_e are given by (3.179)-(3.182). One may immediately obtain that problem (3.183)-(3.188) becomes

$$\min\left((x-x_e)^2+(y-y_e)^2\right) \tag{3.189}$$

under the conditions

$$x \geq 0 \tag{3.190}$$
$$y \geq 0 \tag{3.191}$$
$$x+y \leq 3u_e-3l_e-\frac{1}{2}x_e-\frac{1}{2}y_e. \tag{3.192}$$

Moreover we have

$$l=l_e-\frac{1}{6}(x-x_e) \tag{3.193}$$

and

$$u=u_e+\frac{1}{6}(y-y_e). \tag{3.194}$$

Let us consider the set

$$M_A=\left\{(x,y)\in\mathbb{R}^2 : x\geq 0, y\geq 0, x+y\leq 3u_e-3l_e-\frac{1}{2}x_e-\frac{1}{2}y_e\right\} \tag{3.195}$$

and let us denote by $P_M(Z)$ the orthogonal projection of $Z\in\mathbb{R}^2$ on non-empty set $M\subset\mathbb{R}^2$ with respect to the Euclidean metric on $\mathbb{R}^2$.

It is shown in [25] (Theorem 5) that problem (3.183)-(3.188) has a unique solution.

As a conclusion, $T(A)=[l_T,u_T,x_T,y_T]$ is the nearest trapezoidal fuzzy number to a given fuzzy number A such that (3.176) and (3.177) hold if and only if (x_T,y_T) is the orthogonal projection of (x_e,y_e) on M_A and

$$l_T=l_e-\frac{1}{6}(x-x_e) \tag{3.196}$$
$$u_T=u_e+\frac{1}{6}(y-y_e). \tag{3.197}$$

Since $x_e \geq 0$ and $y_e \geq 0$, we may obtain the following possible orthogonal projections $P_{M_A}(Z_e)$ of $Z_e = (x_e, y_e)$ on M_A:

(i) if $(x_e, y_e) \in M_A$, i.e. $x_e + y_e \leq 3u_e - 3l_e - \frac{1}{2}x_e - \frac{1}{2}y_e$, then we get $x_e + y_e \leq 2u_e - 2l_e$, so $P_{M_A}(x_e, y_e) = (x_e, y_e)$ and hence $x_T = x_e$ and $y_T = y_e$.
(ii) if $\frac{3}{2}x_e - \frac{1}{2}y_e - 3u_e + 3l_e > 0$ then $P_{M_A}(x_e, y_e) = (3u_e - 3l_e - \frac{3}{2}x_e - \frac{1}{2}y_e, 0)$ and thus $x_T = 3u_e - 3l_e - \frac{1}{2}x_e - \frac{1}{2}y_e$ and $y_T = 0$.
(iii) if $\frac{1}{2}x_e - \frac{3}{2}y_e + 3u_e - 3l_e < 0$ then $P_{M_A}(x_e, y_e) = (0, 3u_e - 3l_e - \frac{3}{2}x_e - \frac{1}{2}y_e)$ and hence $x_T = 0$ and $y_T = 3u_e - 3l_e - \frac{1}{2}x_e - \frac{1}{2}y_e$.
(iv) otherwise, i.e. if $x_e + y_e > 2u_e - 2l_e$, $\frac{3}{2}x_e - \frac{1}{2}y_e - 3u_e + 3l_e \leq 0$, $\frac{1}{2}x_e - \frac{3}{2}y_e + 3u_e - 3l_e \geq 0$, then (x_T, y_T) is the orthogonal projection of (x_e, y_e) on the line $x + y = 3u_e - 3l_e - \frac{1}{2}x_e - \frac{1}{2}y_e$ and therefore $x_T = \frac{3}{2}u_e - \frac{3}{2}l_e + \frac{1}{4}x_e - \frac{3}{4}y_e$ and $y_T = \frac{3}{2}u_e - \frac{3}{2}l_e - \frac{3}{4}x_e + \frac{1}{4}y_e$.

Now, by (3.179)-(3.182) and (3.196)-(3.197), we get the following theorem (see [25], Theorem 7).

Theorem 3.18. *For any fuzzy number A the trapezoidal fuzzy number* $T(A) = [l_T, u_T, x_T, y_T]$ *nearest to A with respect to* (1.40)*, which preserves the value and ambiguity, is given as follows*

(a) if $\int_0^1 (3\alpha - 1)A_L(\alpha)d\alpha - \int_0^1 (3\alpha - 1)A_U(\alpha)d\alpha \leq 0$ *then*

$$l_T = \int_0^1 A_L(\alpha)d\alpha \tag{3.198}$$

$$u_T = \int_0^1 A_U(\alpha)d\alpha \tag{3.199}$$

$$x_T = 6\int_0^1 (2\alpha - 1)A_L(\alpha)d\alpha \tag{3.200}$$

$$y_T = -6\int_0^1 (2\alpha - 1)A_U(\alpha)d\alpha, \tag{3.201}$$

(b) if $\int_0^1 (3\alpha - 1)A_L(\alpha)d\alpha - \int_0^1 (3\alpha - 1)A_U(\alpha)d\alpha > 0$, $\int_0^1 (3\alpha - 1)A_L(\alpha)d\alpha + \int_0^1 (\alpha - 1)A_U(\alpha)d\alpha \leq 0$ *and* $\int_0^1 (\alpha - 1)A_L(\alpha)d\alpha + \int_0^1 (3\alpha - 1)A_U(\alpha)d\alpha \geq 0$ *then*

$$l_T = \frac{1}{2}\int_0^1 (3\alpha + 1)A_L(\alpha)d\alpha - \frac{1}{2}\int_0^1 (3\alpha - 1)A_U(\alpha)d\alpha \tag{3.202}$$

$$u_T = -\frac{1}{2}\int_0^1 (3\alpha - 1)A_L(\alpha)d\alpha + \frac{1}{2}\int_0^1 (3\alpha + 1)A_U(\alpha)d\alpha \tag{3.203}$$

$$x_T = 3\int_0^1 (\alpha - 1)A_L(\alpha)d\alpha + 3\int_0^1 (3\alpha - 1)A_U(\alpha)d\alpha \tag{3.204}$$

$$y_T = -3\int_0^1 (3\alpha - 1)A_L(\alpha)d\alpha - 3\int_0^1 (\alpha - 1)A_U(\alpha)d\alpha. \tag{3.205}$$

(c) *if* $\int_0^1(3\alpha-1)A_L(\alpha)d\alpha+\int_0^1(\alpha-1)A_U(\alpha)d\alpha>0$ *then*

$$l_T = 3\int_0^1 \alpha A_L(\alpha)d\alpha - \int_0^1 \alpha A_U(\alpha)d\alpha \tag{3.206}$$

$$u_T = 2\int_0^1 \alpha A_U(\alpha)d\alpha \tag{3.207}$$

$$x_T = -6\int_0^1 \alpha A_L(\alpha)d\alpha + 6\int_0^1 \alpha A_U(\alpha)d\alpha \tag{3.208}$$

$$y_T = 0, \tag{3.209}$$

(d) *if* $\int_0^1(\alpha-1)A_L(\alpha)d\alpha+\int_0^1(3\alpha-1)A_U(\alpha)d\alpha<0$ *then*

$$l_T = 2\int_0^1 \alpha A_L(\alpha)d\alpha \tag{3.210}$$

$$u_T = -\int_0^1 \alpha A_L(\alpha)d\alpha + 3\int_0^1 \alpha A_U(\alpha)d\alpha \tag{3.211}$$

$$x_T = 0 \tag{3.212}$$

$$y_T = -6\int_0^1 \alpha A_L(\alpha)d\alpha + 6\int_0^1 \alpha A_U(\alpha)d\alpha, \tag{3.213}$$

By (1.55)-(1.58) we can express the formula for $T(A)=[l_T,u_T,x_T,y_T]$ given in Theorem 3.18 in the traditional notation, i.e. $T(A)=(t_1,t_2,t_3,t_4)$. Moreover, using expressions for such characteristics of a fuzzy number like its width (1.82), left-hand ambiguity (1.80) and right-hand ambiguity (1.81), we can rewrite the conditions that appear in the above theorem in a way more suitable for interpretation.

Corollary 3.6. *The nearest trapezoidal approximation operator preserving the value and ambiguity is such operator* $T:\mathbb{F}(\mathbb{R})\to\mathbb{F}^T(\mathbb{R})$ *which for any fuzzy number* A *assigns the trapezoidal fuzzy number* $T(A)=T(t_1,t_2,t_3,t_4)$ *as follows*

(a) *if* $Amb(A)\ge\frac{1}{3}w(A)$ *then*

$$t_1 = -6\int_0^1 \alpha A_L(\alpha)d\alpha + 4\int_0^1 A_L(\alpha)d\alpha, \tag{3.214}$$

$$t_2 = 6\int_0^1 \alpha A_L(\alpha)d\alpha - 2\int_0^1 A_L(\alpha)d\alpha, \tag{3.215}$$

$$t_3 = 6\int_0^1 \alpha A_U(\alpha)d\alpha - 2\int_0^1 A_U(\alpha)d\alpha, \tag{3.216}$$

$$t_4 = -6\int_0^1 \alpha A_U(\alpha)d\alpha + 4\int_0^1 A_U(\alpha)d\alpha; \tag{3.217}$$

(b) *if* $Amb(A)<\frac{1}{3}w(A)$ *and* $\frac{1}{3}Amb_L(A)\le Amb_U(A)\le 3Amb_L(A)$ *then*

$$t_1 = 2\int_0^1 A_L(\alpha)d\alpha + 2\int_0^1 A_U(\alpha)d\alpha - 6\int_0^1 \alpha A_U(\alpha)d\alpha, \tag{3.218}$$

$$t_2 = t_3 = 3\int_0^1 \alpha A_L(\alpha)d\alpha + 3\int_0^1 \alpha A_U(\alpha)d\alpha - \int_0^1 A_L(\alpha)d\alpha - \int_0^1 A_U(\alpha)d\alpha, \tag{3.219}$$

$$t_4 = 2\int_0^1 A_L(\alpha)d\alpha + 2\int_0^1 A_U(\alpha)d\alpha - 6\int_0^1 \alpha A_L(\alpha)d\alpha; \tag{3.220}$$

(c) *if* $Amb(A) < \frac{1}{3}w(A)$ *and* $Amb_U(A) < \frac{1}{3}Amb_L(A)$ *then*

$$t_1 = t_2 = t_3 = 2\int_0^1 \alpha A_L(\alpha)d\alpha, \tag{3.221}$$

$$t_4 = 6\int_0^1 \alpha A_U(\alpha)d\alpha - 4\int_0^1 \alpha A_L(\alpha)d\alpha; \tag{3.222}$$

(d) *if* $Amb(A) < \frac{1}{3}w(A)$ *and* $Amb_U(A) > 3Amb_L(A)$ *then*

$$t_1 = 6\int_0^1 \alpha A_L(\alpha)d\alpha - 4\int_0^1 \alpha A_U(\alpha)d\alpha, \tag{3.223}$$

$$t_2 = t_3 = t_4 = 2\int_0^1 \alpha A_U(\alpha)d\alpha. \tag{3.224}$$

Let us notice that, similarly as in Theorem 3.8, the desired operator may assume four possible forms depending on shape of the original fuzzy number A. If the input is rather vague and quite dispersed, i.e. if

$$Amb(A) \geq \frac{1}{3}w(A) \tag{3.225}$$

holds, then $T(A)$ is a trapezoidal but not triangular fuzzy number identical as in Theorem 3.8. Actually, t_1, t_2, t_3, t_4 given by (3.214)-(3.217) are the same as in (3.50)-(3.53).

Otherwise, i.e. if condition (3.225) is not fulfilled since A is less vague (and less dispersed), we obtain a triangular output $T(A)$. It is worth stressing that this is identically as in the case of the nearest trapezoidal approximation preserving the expected value considered in Section 3.5.1. Moreover, similarly as in that case we may get here three possible solutions depending on the symmetry of A: for a symmetrical fuzzy number A or for A with a slight asymmetry its approximation $T(A)$ is given by (3.218)-(3.220); if the left-hand ambiguity is much bigger than the right-hand ambiguity then $T(A)$ is given by (3.221)-(3.222) and conversely, if the right-hand ambiguity is much bigger than the left-hand ambiguity then $T(A)$ is given by (3.223)-(3.224).

To sum up, the nearest trapezoidal approximation preserving the value and the ambiguity might be determined using the following algorithm:

Algorithm 3 *For any $A \in \mathbb{F}(\mathbb{R})$*

Step 1. *If $Amb(A) \geq \frac{1}{3}w(A)$ then apply operator T_1 given by* (3.214)-(3.217)*, else*
Step 2. *if $\frac{1}{3}Amb_L(A) \leq Amb_U(A) \leq 3Amb_L(A)$ then apply operator T_2 given by* (3.218)-(3.220)*, else*
Step 3. *if $Amb_U(A) < \frac{1}{3}Amb_L(A)$ then apply operator T_3 given by* (3.221)-(3.222)*, else*
Step 4. *apply operator T_4 given by* (3.223)-(3.224).

Let us also discuss briefly main properties of the nearest trapezoidal approximation preserving the value and the ambiguity. Such properties like the translation invariance, scale invariance or identity (see Section 3.2) are immediate.

The continuity of the operator T is an immediate consequence of the following Lipschitz property:

Theorem 3.19. *For any fuzzy numbers A and B the nearest trapezoidal approximation operator T preserving the ambiguity and value satisfies*

$$d(T_e(A), T_e(B)) \leq (2\sqrt{2}+1)d(A,B). \tag{3.226}$$

For the proof we refer the reader to [25], Theorem 19. The problem of finding the best Lipschitz constant of the trapezoidal approximation operator given in Theorem 3.18 is not easy to study (for the deeper discussion we also refer the reader to [25]).

As we have mention above the nearest trapezoidal approximation operator preserving the expected interval, considered in Section 3.5.1, does not preserve - in general - the value and ambiguity. Actually, according to Theorem 3.9, we have invariance of both characteristics only for more dispersed fuzzy numbers. In all other cases the ambiguity of the output exceeds the ambiguity of the input. Moreover, the value of the output depends on the symmetry of the input, i.e. it may remain invariant for the inputs neither too skew to the left nor to the right or be smaller or bigger than $Val(A)$ depending on whether A is is relatively skew to the left or to the right. In all cases the expected value and the width are invariant under such approximation since it is guaranteed by the invariance of the expected interval. Indeed, the expected value is the middle point of the expected interval while the width is equal to its length.

Keeping this in mind one may ask about the behavior of this two characteristics under the nearest trapezoidal approximation operator preserving the value and ambiguity. Since by Theorem 3.18 and Corollary 3.6 our operator may assume different forms depending on the shape of the input fuzzy number A, we have to consider each case separately.

Firstly, we may conclude that for those fuzzy numbers that satisfy (3.225) we get the invariance of the expected value and of the width since for such fuzzy numbers - as it was stated above - the nearest trapezoidal approximation operator preserving the value and ambiguity behaves identically as the nearest trapezoidal approximation operator preserving the expected interval.

In all other cases, i.e. if condition (3.225) is not satisfied, we may compute the width of the output using appropriate formulae from Theorem 3.18. By (1.82) and (1.51)-(1.52) we get

$$w(T(A)) = u_T - l_T.$$

Please note, that in three cases (b)-(d) considered in Theorem 3.18 we obtain the same value of $w(T(A))$, i.e.

$$w(T(A)) = 3\left(\int_0^1 \alpha A_U(\alpha)d\alpha - \int_0^1 \alpha A_L(\alpha)d\alpha\right) = 3Amb(A).$$

However, by Corollary 3.6, in these three cases $Amb(A) < \frac{1}{3}w(A)$, hence we obtain $w(T(A)) < w(A)$.

By (1.72) and (1.51)-(1.52) we get the formula for the expected value of the output

$$EV(T(A)) = \frac{1}{2}(l_T + u_T). \tag{3.227}$$

Substituting (3.202) and (3.203) into (3.227) we get

$$EV(T(A)) = \frac{1}{2}\left(\int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha\right) = EV(A).$$

Now, let us consider case (c) and substitute (3.206) and (3.207) into (3.227). We get

$$EV(T(A)) = \frac{1}{2}\left(3\int_0^1 \alpha A_L(\alpha)d\alpha + \int_0^1 \alpha A_U(\alpha)d\alpha\right).$$

However, since in this case $\int_0^1 (3\alpha - 1)A_L(\alpha)d\alpha + \int_0^1 (\alpha - 1)A_U(\alpha)d\alpha > 0$, therefore $3\int_0^1 \alpha A_L(\alpha)d\alpha + \int_0^1 \alpha A_U(\alpha)d\alpha > \int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha$. Thus, $EV(T(A)) > \frac{1}{2}(\int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha) = EV(A)$.

Finally, substituting (3.210) and (3.211) into (3.227) we get

$$EV(T(A)) = \frac{1}{2}\left(\int_0^1 \alpha A_L(\alpha)d\alpha + 3\int_0^1 \alpha A_U(\alpha)d\alpha\right),$$

but since in case (d) the required condition is $\int_0^1 (\alpha - 1)A_L(\alpha)d\alpha + \int_0^1 (3\alpha - 1)A_U(\alpha)d\alpha < 0$, thus we have $\int_0^1 \alpha A_L(\alpha)d\alpha + 3\int_0^1 \alpha A_U(\alpha)d\alpha < \int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha$. Therefore, $EV(T(A)) < \frac{1}{2}(\int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha) = EV(A)$.

This way we have proved the following theorem.

Theorem 3.20. *Let $A \in \mathbb{F}(\mathbb{R})$ and $T : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^T(\mathbb{R})$ denote the nearest trapezoidal approximation operators preserving the value and ambiguity. Then:*

(a) *if $Amb(A) \geq \frac{1}{3}w(A)$ then*

$$EV(T(A)) = EV(A),$$
$$w(T(A)) = w(A),$$

(b) *if* $Amb(A) < \frac{1}{3}w(A)$ *and* $\frac{1}{3}Amb_L(A) \leq Amb_U(A) \leq 3Amb_L(A)$ *then*

$$EV(T(A)) = EV(A),$$
$$w(T(A)) < w(A),$$

(c) *if* $Amb(A) < \frac{1}{3}w(A)$ *and* $Amb_U(A) < \frac{1}{3}Amb_L(A)$ *then*

$$EV(T(A)) > EV(A),$$
$$w(T(A)) < w(A),$$

(d) *if* $Amb(A) < \frac{1}{3}w(A)$ *and* $Amb_U(A) > 3Amb_L(A)$ *then*

$$EV(T(A)) < EV(A),$$
$$w(T(A)) < w(A).$$

One may easily notice that Theorem 3.20 is somehow dual to Theorem 3.9.

3.6 Nearest piecewise linear approximation of fuzzy numbers

Trapezoidal approximation of fuzzy numbers delivers the outputs with the simplest possible membership function acquired by linear sides. Approximation of a fuzzy number by the closest trapezoidal one does not guarantee automatically any other interesting properties. Therefore, we often look for the approximation that has some additional properties like the invariance of the expected interval, discussed in Section 3.5.

It seems that the core and the support belong to the most important characteristics of fuzzy numbers. It is quite obvious since these very sets are the only ones which are connected with our "sure" knowledge. Actually, the core contains all the points which surely belong to the fuzzy set under study. On the other hand, the complement of the support consists of the points that surely do not belong to given fuzzy set. The belongingness of all other points to the fuzzy set under discussion is just a matter of degree described quantitatively by the membership function. Hence, one may easily agree that both the support and core play a key role in fuzzy set analysis. However, if we try to approximate a fuzzy number by a trapezoidal one that preserves both the support and core of the input, the approximation problem simplifies too much since we obtain the unique solution just by joining the borders of the support and core by the straight lines. Unfortunately, the output of such approximation may be significantly distant from the input. The way out from this dilemma is to consider the approximation by a trapezoidal fuzzy number which is as close as possible to the input and preserves either the core or the support. However, one may easily indicate examples where the output of the approximation with fixed core has the support significantly different than the support of the input. And conversely, the output of the approximation with fixed support may have the core significantly different than the core of the input. A possible way-out from that dilemma is to consider the trapezoidal approximation with restrictions on support and core discussed in Section 3.5.2. Unfortunately, the solution given there does not guarantee the preservation of the support and core but some relations between support/core input and the output expressed by appropriate inequalities only.

This discussion shows that usually we cannot obtain a satisfying trapezoidal approximation of an arbitrary fuzzy number that fulfills the nearness criterion and preserves both the support and core. In this paper we propose to consider the 1-knot piecewise linear fuzzy numbers (described in Section 1.3.8) as a reasonable solution of the approximation problem satisfying requirements. More precisely, we suggest to approximate a fuzzy number by the closest piecewise linear 1-knot fuzzy number having the same core and the same support as the input.

Let us consider any fuzzy number $A \in \mathbb{F}(\mathbb{R})$. Suppose we want to approximate A by an α_0-piecewise linear 1-knot fuzzy number S. Our goal now is to find the approximation which fulfills the following requirements:

1. Indicate the optimal knot α_0 for the piecewise linear 1-knot fuzzy number approximation of A, i.e. we are looking for the solution $S(A)$ in $\mathbb{F}^{\pi[0,1]}(\mathbb{R})$.
2. The solution should fulfill the so-called nearness criterion, i.e. for any fuzzy number A the solution $S(A)$ should be the α_0-piecewise linear 1-knot fuzzy number nearest to A with respect to some predetermined metric. In our case we consider the distance d given by (1.40).
3. The solution should preserve the core and the support of A.

More formally, we are looking for such $S^* = S^*(A) \in \mathbb{F}^{\pi[0,1]}(\mathbb{R})$ that

$$d(A,S^*) = \min_{S\in\mathbb{F}^{\pi[0,1]}(\mathbb{R})} d(A,S), \tag{3.228}$$

which satisfies the following constraints:

$$\text{core}(S^*) = \text{core}(A), \tag{3.229}$$

$$\text{supp}(S^*) = \text{supp}(A). \tag{3.230}$$

At first, let us investigate whether the above problem has at least one solution for every $A \in \mathbb{F}(\mathbb{R})$. For that we will use the property that the space $(F(\mathbb{R}), d, +, \cdot)$ can be embedded in the Hilbert space $\left(L^2[0,1] \times L^2[0,1], \widetilde{d}, \oplus, \odot\right)$ (see e.g. [76]). Therefore, we have $d(A,B) = \widetilde{d}(A,B)$, $A+B = A \oplus B$ and $\lambda \cdot A = \lambda \odot A$, for all $A, B \in \mathbb{F}(\mathbb{R})$ and $\lambda \in [0,\infty)$. By Proposition 4 in [76] it is known that $\mathbb{F}^{\pi[0,1]}(\mathbb{R})$ is a closed subset of $L^2[0,1] \times L^2[0,1]$ in the topology generated by $\widetilde{d}$. Unfortunately, it may happen that the set

$$CS(A) = \left\{S \in \mathbb{F}^{\pi[0,1]}(\mathbb{R}) : \text{core}(S) = \text{core}(A), \text{supp}(S) = \text{supp}(A)\right\} \tag{3.231}$$

would not be closed in $\mathbb{F}^{\pi[0,1]}(\mathbb{R})$. Indeed, suppose that $A_\beta = [\beta^3, 1]$, $\beta \in [0,1]$. Then let us consider a sequence $(\mathrm{S}(\alpha_n, \mathbf{s}_n))_{n\geq 1}$, $\mathbf{s}_n = (s_{n,1}, ..., s_{n,6})$ in $CS(A)$, where for each $n \geq 1$ we have $\alpha_n = (n-1)/n$, $s_{n,1} = s_{n,2} = 0$ and $s_{n,3} = ... = s_{n,6} = 1$. It is immediate that $(\widetilde{d}) \lim_{n\to\infty} \mathrm{S}(\alpha_n, \mathbf{s}_n) = (d) \lim_{n\to\infty} \mathrm{S}(\alpha_n, \mathbf{s}_n) = [0,1]$ and since $\mathrm{core}([0,1]) \neq \mathrm{core}(A)$ it results that the set $CS(A)$ is not closed in $L^2[0,1] \times L^2[0,1]$, nor in $F(\mathbb{R})$. Therefore, it is an open question whether problem (3.228)-(3.230) has a solution for any $A \in \mathbb{F}(\mathbb{R})$.

Interestingly, the solution always exists if we consider a local approximation problem. Suppose that $0 < a < b < 1$ and let us consider the set $\mathbb{F}^{\pi[a,b]}(\mathbb{R}) = \{\mathrm{S}(\alpha, \mathbf{s}) \in \mathbb{F}^{\pi[0,1]}(\mathbb{R}) : a \leq \alpha \leq b\}$. Now let us consider the following set

$$CS_{a,b}(A) = \left\{ S \in \mathbb{F}^{\pi[a,b]}(\mathbb{R}) : \mathrm{core}(S) = \mathrm{core}(A), \mathrm{supp}(S) = \mathrm{supp}(A) \right\}. \quad (3.232)$$

We are looking for such $S^* = S^*(A) \in CS_{a,b}(A)$ that

$$d(A, S^*) = \min_{S \in CS_{a,b}(A)} d(A, S). \quad (3.233)$$

Obviously, there is a sequence $(\mathrm{S}(\alpha_n, \mathbf{s}_n))_{n\geq 1}$, in $CS_{a,b}(A)$, such that

$$\lim_{n\to\infty} d(A, \mathrm{S}(\alpha_n, \mathbf{s}_n)) = \inf_{S \in CS_{a,b}(A)} d(A, S) := m. \quad (3.234)$$

Let $n_0 \in \mathbb{N}$ be such that $d(A, \mathrm{S}(\alpha_n, \mathbf{s}_n)) \leq m + 1$ for all $n \geq n_0$. This implies that $d(0, \mathrm{S}(\alpha_n, \mathbf{s}_n)) \leq d(0, A) + d(A, \mathrm{S}(\alpha_n, \mathbf{s}_n)) \leq d(0, A) + m + 1$ for all $n \geq n_0$. Therefore, the sequence $(\mathrm{S}(\alpha_n, \mathbf{s}_n))_{n\geq 1}$ is bounded with respect to metric d and hence with respect to $\widetilde{d}$. By Lemma 2 (iii) in [76] it results that each sequence $(c_{n,i})_{n\geq 1}$, $i = 1, \ldots, 8$, is bounded, where

$$\begin{aligned} s_{n,1} &= c_{n,1}, & s_{n,2} &= c_{n,2} \cdot \alpha_n + c_{n,1}, & s_{n,3} &= c_{n,3} + c_{n,4}, \\ s_{n,4} &= c_{n,7} + c_{n,8}, & s_{n,5} &= c_{n,5} + c_{n,6} \cdot \alpha_n, & s_{n,6} &= c_{n,5}. \end{aligned}$$

Without loss of generality let us suppose that $\lim_{n\to\infty} \alpha_n = \alpha_0$ (obviously we have $\alpha_0 \in [a,b]$) and $\lim_{n\to\infty} c_{n,i} = c_i$, $i = 1, \ldots, 8$. Letting $n \to \infty$ in the above equations and denoting $\mathbf{s} = (s_1, ..., s_6)$, where $s_i = \lim_{n\to\infty} s_{n,i}$, $i = 1, \ldots, 6$, it easily results that $\mathrm{S}(\alpha_0, \mathbf{s}) \in \mathbb{F}^{\pi[a,b]}(\mathbb{R})$. Then, since $\mathrm{S}(\alpha_n, \mathbf{s}_n) \in CS_{a,b}(A)$ for all $n \geq 1$, it follows that $s_{n,1} = A_L(0)$, $s_{n,3} = A_L(1)$, $s_{n,4} = A_U(1)$ and $s_{n,6} = A_U(0)$ and therefore we easily obtain that $\mathrm{S}(\alpha_0, \mathbf{s})$ preserves the core and support of A and hence $\mathrm{S}(\alpha_0, \mathbf{s}) \in CS_{a,b}(A)$. On the other hand, by Lemma 3 in [76] (making some suitable substitutions) we also obtain that $(\widetilde{d}) \lim_{n\to\infty} \mathrm{S}(\alpha_n, \mathbf{s}_n) = \mathrm{S}(\alpha_0, \mathbf{s})$. This property together with relation (3.234) and the continuity of d, implies that $d(A, \mathrm{S}(\alpha_0, \mathbf{s})) = m$. Hence we have just proved that problem (3.233) has at least one solution. Note that one can easily prove that $CS_{a,b}(A)$ is not convex in $L^2[0,1] \times L^2[0,1]$ which means that the solution of problem (3.233) may not be unique. The results are summarized in the following theorem.

Theorem 3.21. *([77], Theorem 1) If* $A \in \mathbb{F}(\mathbb{R})$ *and* $0 < a < b < 1$*, then there exists at least one element* $S^* = S^*(A) \in \mathbb{F}^{\pi[a,b]}(\mathbb{R})$ *such that* $d(A, S^*) = \min_{S \in CS_{a,b}(A)} d(A, S)$.

Let us show how to find a solution to problem (3.233). We have to minimize the function

$$\begin{aligned} f(\alpha,x,y) = & \int_0^\alpha \left(A_L(\beta) - \left(A_L(0) + (x - A_L(0)) \cdot \frac{\beta}{\alpha}\right)\right)^2 d\beta \\ & + \int_\alpha^1 \left(A_L(\beta) - \left(x + (A_L(1) - x) \cdot \frac{\beta - \alpha}{1 - \alpha}\right)\right)^2 d\beta \\ & + \int_0^\alpha \left(A_U(\beta) - \left(y + (A_U(0) - y) \cdot \frac{\alpha - \beta}{\alpha}\right)\right)^2 d\beta \\ & + \int_\alpha^1 \left(A_U(\beta) - \left(A_U(1) + (y - A_U(1)) \cdot \frac{1 - \beta}{1 - \alpha}\right)\right)^2 d\beta \end{aligned}$$

subject to $A_L(0) \le x \le A_L(1)$ and $A_U(1) \le y \le A_U(0)$.

This problem may have more than one solution and, in addition, it seems to be difficult to be solved analytically in this form since the equation $f'_\alpha(\alpha,x,y) = 0$ cannot be solved in general as we are forced to work with functions where we cannot separate α from the integral. Therefore, we will start by considering the knot $\alpha = \alpha_0$ being fixed. For some $\alpha_0 \in (0,1)$ we want to minimize the function $g_{\alpha_0}(x,y) = f(\alpha_0,x,y)$ with the same restrictions as above. Obviously we can split this problem into two independent sub-problems. Firstly, we have to minimize the function

$$\begin{aligned} x \mapsto & \int_0^{\alpha_0} \left(A_L(\beta) - \left(A_L(0) + (x - A_L(0)) \cdot \frac{\beta}{\alpha_0}\right)\right)^2 d\beta \\ & + \int_{\alpha_0}^1 \left(A_L(\beta) - \left(x + (A_L(1) - x) \cdot \frac{\beta - \alpha_0}{1 - \alpha_0}\right)\right)^2 d\beta \end{aligned}$$

on the interval $[A_L(0), A_L(1)]$ and then we have to minimize the function

$$\begin{aligned} y \mapsto & \int_0^{\alpha_0} \left(A_U(\beta) - \left(y + (A_U(0) - y) \cdot \frac{\alpha_0 - \beta}{\alpha_0}\right)\right)^2 d\beta \\ & + \int_{\alpha_0}^1 \left(A_U(\beta) - \left(A_U(1) + (y - A_U(1)) \cdot \frac{1 - \beta}{1 - \alpha_0}\right)\right)^2 d\beta \end{aligned}$$

on the interval $[A_U(1), A_U(0)]$. Obviously, the above functions are quadratic functions of one variable and after some simple calculations we obtain their unique minimum points on $\mathbb{R}$ as

$$x_m = 3\int_0^{\alpha_0}\left(A_L(\beta)-A_L(0)\cdot\frac{\alpha_0-\beta}{\alpha_0}\right)\cdot\frac{\beta}{\alpha_0}d\beta$$
$$+3\int_{\alpha_0}^{1}\left(A_L(\beta)-A_L(1)\cdot\frac{\beta-\alpha_0}{1-\alpha_0}\right)\cdot\frac{1-\beta}{1-\alpha_0}d\beta$$

and

$$y_m = 3\int_0^{\alpha_0}\left(A_U(\beta)-A_U(0)\cdot\frac{\alpha_0-\beta}{\alpha_0}\right)\cdot\frac{\beta}{\alpha_0}d\beta$$
$$+3\int_{\alpha_0}^{1}\left(A_U(\beta)-A_U(1)\cdot\frac{\beta-\alpha_0}{1-\alpha_0}\right)\cdot\frac{1-\beta}{1-\alpha_0}d\beta.$$

From here we easily obtain the solutions of our two sub-problems as

$$x_0 = \begin{cases} A_L(0) & \text{if} \quad x_m < A_L(0), \\ A_L(1) & \text{if} \quad x_m > A_L(1), \\ x_m & \text{if} \quad A_L(0) \le x_m \le A_L(1) \end{cases} \tag{3.235}$$

and

$$y_0 = \begin{cases} A_U(1) & \text{if} \quad y_m < A_U(1), \\ A_U(0) & \text{if} \quad y_m > A_U(0), \\ y_m & \text{if} \quad A_U(1) \le y_m \le A_U(0). \end{cases} \tag{3.236}$$

When a computer implementation is needed, in most of the cases, x_m and y_m may be easily calculated via numeric integration (see [96]).

Thus we have just proved for fixed α the existence and uniqueness of the piecewise linear 1-knot approximation which preserves the core and the support. More precisely, we have the following approximation result.

Theorem 3.22. *([77], Theorem 2) Suppose that* $\alpha_0 \in (0,1)$ *and for some fuzzy number A let us define the set*

$$CS_{\alpha_0}(A) = \{S \in \mathbb{F}^{\pi(\alpha_0)}(\mathbb{R}) : \text{core}(S) = \text{core}(A) \textit{ and } \text{supp}(S) = \text{supp}(A)\}.$$

Then there exists a unique best approximation (with respect to metric d) of A relatively to the set $CS_{\alpha_0}(A)$*. This approximation is* $S_{\alpha_0}(A) = \mathbf{S}(\alpha_0, \mathbf{s}(A))$*,* $\mathbf{s}(A) = (s_1(A), ..., s_6(A))$*, where*

$$s_1(A) = A_L(0), \quad s_2(A) = x_0, \quad s_3(A) = A_L(1),$$
$$s_4(A) = A_U(1), \quad s_5(A) = y_0, \quad s_6(A) = A_U(0),$$

and x_0, y_0 *are given by (3.235) and (3.236) respectively.*

To sum up, we use the previous theorem to approach a solution $S^*(A) \in CS_{a,b}(A)$ of problem (3.233). We construct a sequence $(S_{\alpha_n}(A))_{n\ge 1}$ in $CS_{a,b}(A)$ such that $(d)\lim_{n\to\infty} S_{\alpha_n}(A) = S^*(A)$. Here, $S_{\alpha_n}(A)$ is the unique best approximation of A relatively to the set $CS_{\alpha_n}(A)$.

It is worth noticing that when using numerical optimization techniques, one can get a local minimum which is not its global minimum, so an algorithm falls not into optimal but a suboptimal solution. For more details and examples we refer the reader to [77].

Finally let us mention that a general case of the piecewise linear 1-knot fuzzy number approximation is broadly discussed in [76]. We consider there an approximation without restrictions on the support and core both with a fixed knot and the problem of the optimal choice of the knot of the piecewise linear fuzzy number.

3.7 Nonlinear approximations of fuzzy numbers

3.7.1 Basic ideas and tools

Trapezoidal approximation of fuzzy numbers plays a fundamental role in many fields and applications where extended or massive fuzzy computations are required. As we have mentioned above, the main advantage of the trapezoidal approximation is its simplicity in calculations, natural interpretation and satisfying properties. However, it seems that in some situations a nonlinear approximation of fuzzy numbers would be desirable. The piecewise linear approximation of fuzzy numbers described in Section 3.6 may be perceived as an idea of extending possible outputs still remaining in a family of membership functions having simple shapes. Although it is a quite intuitive solution, its shortcoming is nondifferentiation of its sides. Some suggestions of other approaches to nonlinear approximation of fuzzy numbers can be found e.g. in [1, 2, 20, 21, 158]. In this section we present a general framework for the nonlinear approximation of fuzzy numbers satisfying the following key points:

- the basic general model should be able to apply for fuzzy numbers given both in *L-R* and *L-U* representations;
- the families of parameterized monotonic functions to be used are sufficiently flexible to cover a large set of possible curves of membership functions;
- our method should fulfill some approximation criteria, depending usually on the application at hand, such as least squares or other distance minimization, support or core or expected interval preservation, nearest ambiguity or value approximation among others.

To reach this goal we need special families of flexible monotonic curves described by sufficiently large number of parameters to allow multiple approximation criteria and satisfy various requirements. Let us consider the following two families of nonlinear monotonic functions to use as shape generators (see [178]).

(1) the (2,2)-rational standardized monotonic spline

$$p_{R2}(t;\beta_0,\beta_1) = \frac{t^2 + \beta_0 t(1-t)}{1 + (\beta_0 + \beta_1 - 2)t(1-t)}, \tag{3.237}$$

where $\beta_0, \beta_1 \geq 0$ and $t \in [0,1]$;

(2) the mixed cubic-exponential spline

$$p_{MS}(t;\beta_0,\beta_1) = \frac{1}{a}\left[t^2(3-2t)+\beta_0-\beta_0(1-t)^a+\beta_1 t^a\right], \tag{3.238}$$

where $a = 1+\beta_0+\beta_1$, $\beta_0, \beta_1 \geq 0$ and $t \in [0,1]$.

It is easy to verify that for arbitrary nonnegative values of parameters β_0, β_1 the two families of functions are monotonic for all $t \in [0,1]$ and satisfy following conditions (the derivatives $p\prime$ are made with respect to the first variable t):

$$\begin{aligned} p(0) &= 0, \\ p(1) &= 1, \\ p\prime(0) &= \beta_0, \\ p\prime(1) &= \beta_1, \end{aligned}$$

where p denotes either p_{R2} or p_{MS} spline.

All the proposed shape functions are monotonic over $[0,1]$. Note that generally it is not true for standard splines or other polynomials of degree greater than two. Please also note the following properties that might be of interest:

- if $\beta_0 = \beta_1 = 1$ then both p_{R2} and p_{MS} are linear;
- if $\beta_0 + \beta_1 = 2$ then both p_{R2} and p_{MS} are quadratic;
- if $\beta_0 + \beta_1$ is integer (or zero) then p_{MS} is a non-monotonic polynomial.

One may prove the following important lemma.

Lemma 3.7. *([124], Lemma 1) Let p denote either p_{R2} or p_{MS} spline. Then we have*

(i) $0 < \int_0^1 p(t;\beta_0,\beta_1)dt < 1 \quad \forall \beta_0, \beta_1 \geq 0,$

(ii) $\lim_{\beta_1 \to +\infty} \int_0^1 p(t;0,\beta_1)dt = 0,$

(iii) $\lim_{\beta_0 \to +\infty} \int_0^1 p(t;\beta_0,0)dt = 1.$

By Lemma 3.7 both p_{R2} and p_{MS} are able to "cover" the whole square $[0,1] \times [0,1]$ giving an infinite number of shape functions.

Some particular cases of these functions are of interest to meet specific requirements. For example, $p_{R2}(\frac{1}{2};\beta_0,\beta_1) = \frac{1}{2}$ if and only if $\beta_0 = \beta_1 = \beta$ and then $\int_0^1 p_{R2}(t;\beta,\beta)dt = \frac{1}{2}$ for all $\beta \geq 0$.

A family of curves obtained from $p_{R2}(t;\beta_0,\beta_1)$ for different values of parameters is depicted in Figure 3.6.

We can also use other one-parameter flexible forms of the splines discussed above, like $p_{R2,1}(t,a) = p_{R2}(t;a,0)$, i.e.

$$p_{R2,1}(t,a) = \frac{t^2-at^2+at}{1+at-at^2-2t+2t^2}, \tag{3.239}$$

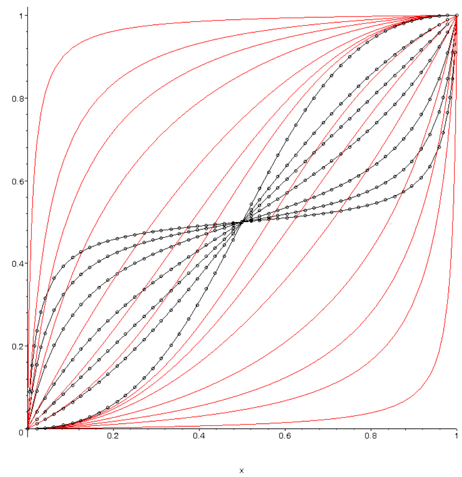

Fig. 3.6 $p_{R2}(t;\beta_0,\beta_1)$ curves for different β_0,β_1 (pointed lines correspond to $\beta_0=\beta_1$).

where $t\in[0,1]$, $a\geq 0$ or $p_{R2,2}(t,b)=p_{R2}(t;0,b)$, i.e.

$$p_{R2,2}(t,b)=\frac{t^2}{1+bt-bt^2-2t+2t^2}, \tag{3.240}$$

where $t\in[0,1]$, $b\geq 0$. Possible shapes obtained for different values of parameters a and b are shown in Figure 3.7 and Figure 3.8, respectively. Note that $p_{R2,1}(t;a)+p_{R2,2}(1-t;a)=1$ for all $t\in[0,1]$.

We can also consider the combinations of two splines, i.e.

$$P_{R2,\lambda}(t;a,b)=(1-\lambda)p_{R2,1}(t;a)+\lambda p_{R2,2}(t;b), \tag{3.241}$$

where $\lambda\in[0,1]$, getting thus three-parameter curves.

Analogous constructions can be obtained by using p_{MS} splines, like $p_{MS,1}(t,a)=p_{MS}(t;a,0)$, i.e.

$$p_{MS,1}(t,a)=\frac{1}{1+a}[t^2(3-2t)+a-a(1-t)^{1+a}], \tag{3.242}$$

where $t\in[0,1]$ and $a\geq 0$, or $p_{MS,2}(t,b)=p_{MS}(t;0,b)$, i.e.

$$p_{MS,2}(t,b)=\frac{1}{1+b}[t^2(3-2t)+bt^{1+b}], \tag{3.243}$$

where $t\in[0,1]$ and $b\geq 0$. We can also combine the two with $\lambda\in[0,1]$ obtaining

$$P_{MS,\lambda}(t;a,b)=(1-\lambda)p_{MS,1}(t;a)+\lambda p_{MS,2}(t;b). \tag{3.244}$$

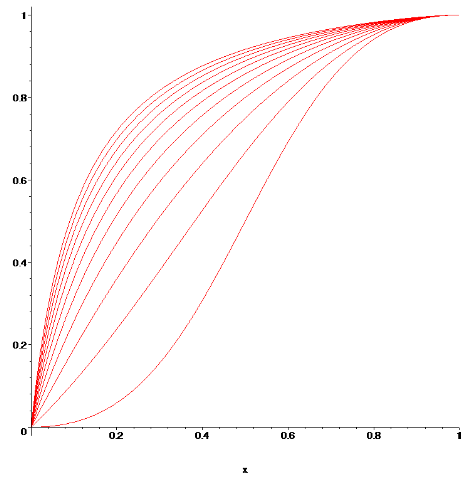

Fig. 3.7 $p_{R2,1}(t;a)$ for different $a \geq 0$.

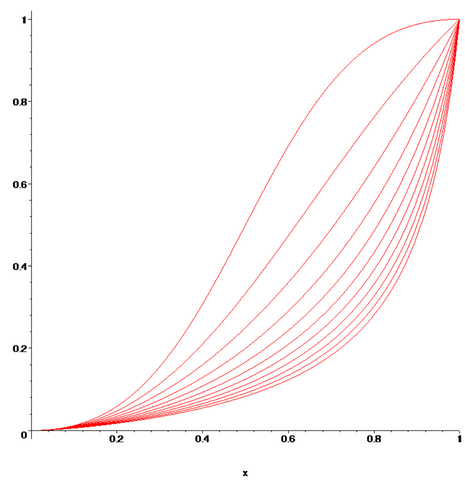

Fig. 3.8 $p_{R2,2}(t;b)$ for different $b \geq 0$.

Examples of curves $P_{MS,0.5}(t;a,b)$ are in Figure 3.9.

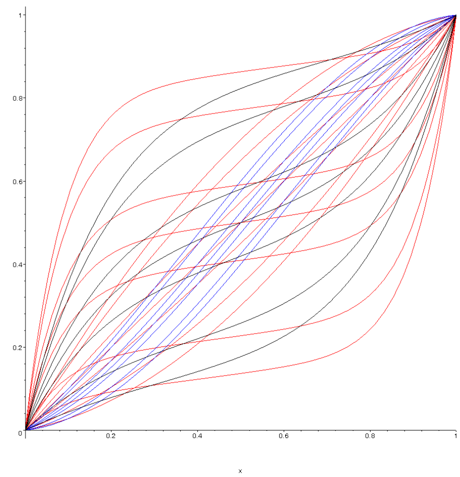

Fig. 3.9 $P_{MS,0.5}(t;a,b)$ for different $a,b \geq 0$.

3.7.2 Splines and fuzzy numbers

We have discussed the parametric functions $p_{R2}(.;\beta_0,\beta_1)$ and $p_{MS}(.;\beta_0,\beta_1)$ because of their capability to construct fuzzy numbers. Actually, if p denotes either p_{R2} or p_{MS} spline then

$$\mu_W(x) = \begin{cases} p(\frac{x-w_1}{w_2-w_1};\beta_{0,L},\beta_{1,L}) & \text{if } x \in [w_1,w_2] \\ 1 & \text{if } x \in [w_2,w_3] \\ 1-p(\frac{x-w_3}{w_4-w_3};\beta_{0,R},\beta_{1,R}) & \text{if } x \in [w_3,w_4] \\ 0 & \text{otherwise} \end{cases} \tag{3.245}$$

is a membership function of a fuzzy number W expressed using the L-R representation. The same fuzzy number can be expressed in the L-U representation by its alpha-cuts $[W_L(\alpha),W_U(\alpha)]$ for $\alpha \in [0,1]$, where

$$W_L(\alpha) = w_1 + (w_2-w_1)p(\alpha;\beta_0^-,\beta_1^-), \tag{3.246}$$

$$W_U(\alpha) = w_4 + (w_3-w_4)p(\alpha;\beta_0^+,\beta_1^+). \tag{3.247}$$

Here, obviously, $w_1 \leq w_2 \leq w_3 \leq w_4$. If needed, we may denote w_1,w_2,w_3 and w_4 also by $W_L(0)$, $W_L(1)$, $W_U(1)$ and $W_U(0)$, respectively.

As it is possible to go from the L-R to the L-U representations by inverting the model functions, further on we will use the L-U form only (however, analogous formulations are possible for the L-R form as well). The obvious relations between the derivatives of $\mu_W(x)$ and the derivatives of $W_L(\alpha)$ and $W_U(\alpha)$ gives (provided they are not null)

$$\beta_{0,L} = \frac{1}{\beta_0^-},\ \beta_{1,L} = \frac{1}{\beta_1^-},\ \beta_{0,R} = \frac{1}{\beta_0^+},\ \beta_{1,R} = \frac{1}{\beta_1^+}. \tag{3.248}$$

A family of all fuzzy numbers with sides modeled by p_{R2} or p_{MS} spline will be denoted by $\mathbb{F}^{p_{R2}}$ or $\mathbb{F}^{p_{MS}}$, respectively (or simply $\mathbb{F}^p$ when we discuss fuzzy numbers modeled by p which is either p_{R2} or p_{MS}).

Let us consider the expected interval $EI(W) = [EI_L(W), EI_U(W)]$ of a fuzzy number $W \in \mathbb{F}^p$. By (1.71) and (3.246)-(3.247) it is given by

$$EI_L(W) = w_1 + (w_2 - w_1)\int_0^1 p(\alpha;\beta_0^-,\beta_1^-)d\alpha, \tag{3.249}$$
$$EI_U(W) = w_4 + (w_3 - w_4)\int_0^1 p(\alpha;\beta_0^+,\beta_1^+)d\alpha$$

and hence we have four parameters $\beta_0^-,\beta_1^-,\beta_0^+,\beta_1^+ \geq 0$ free for our further purposes. Moreover, we may have six parameters of freedom if $P_{R2,\lambda^-}(t;a^-,b^-)$ and $P_{R2,\lambda^+}(t;a^+,b^+)$ or $P_{MS,\lambda^-}(t;a^-,b^-)$ and $P_{MS,\lambda^+}(t;a^+,b^+)$ are used.

If $\beta_0^- = \beta_1^- = \beta^-$ and $\beta_0^+ = \beta_1^+ = \beta^+$ we get $\int_0^1 p_{R2}(t;\beta^\pm,\beta^\pm)dt = \frac{1}{2}$ and the expected interval is

$$EI(W) = \left[\frac{w_1 + w_2}{2}, \frac{w_3 + w_4}{2}\right],$$

for all nonnegative β^-,β^+. Therefore, if we are interested in such approximation of a fuzzy number A that preserves its expected interval then we set the following constraints

$$w_1 + w_2 = 2\int_0^1 A_L(\alpha)d\alpha,$$
$$w_3 + w_4 = 2\int_0^1 A_U(\alpha)d\alpha$$

on $w_1 \leq w_2 \leq w_3 \leq w_4$ while the two parameters $\beta^-,\beta^+ \geq 0$ still can be used for additional requirements. Thus, using our approach we obviously loose the benefits of the approximation with linear sides but the form of $p(t;\beta,\beta)$ is simple and it is easy to invert analytically by solving a quadratic equation instead of linear. It also reduces to linear if $\beta = 1$.

If we like to preserve or approximate the middle set (i.e. the alpha cut for $\alpha = 0.5$) we have

$$p(\frac{1}{2};\beta_0,\beta_1) = \frac{1+\beta_0}{\beta_0 + \beta_1 + 2}$$

and we obtain the following conditions

$$w_1 + (w_2 - w_1)\frac{1+\beta_0^-}{\beta_0^- + \beta_1^- + 2} = A_L(0.5),$$
$$w_4 + (w_3 - w_4)\frac{1+\beta_0^+}{\beta_0^+ + \beta_1^+ + 2} = A_U(0.5),$$

giving (if w_1, w_2, w_3, w_4 are known) the following equations/constraints for the parameter β:

$$(w_2 - A_L(0.5))\beta_0^- + (w_1 - A_L(0.5))\beta_1^- = 2A_L(0.5) - w_1 - w_2,$$
$$(w_3 - A_U(0.5))\beta_0^+ + (w_4 - A_U(0.5))\beta_1^+ = 2A_U(0.5) - w_3 - w_4.$$

In the special case of $\beta_0 + \beta_1 = 2$ we obtain a parabolic shape function

$$p(t;\beta_0,\beta_1) = Q(t;\beta) = t^2 + (2-\beta)t(1-t)$$

where $\beta_0 = 2-\beta, \beta_1 = \beta$ with $\beta \in [0,2]$ and

$$\int_0^1 Q(\alpha;\beta)d\alpha = \frac{1}{3} + \frac{1}{6}\beta$$

is in the range $[\frac{1}{3}, \frac{2}{3}]$. In this case the equations for the expected interval invariance are as follows

$$a_1 + (a_2 - a_1)(\frac{1}{3} + \frac{1}{6}\beta^-) = 2\int_0^1 A_L(\alpha)d\alpha,$$
$$a_4 + (a_3 - a_4)(\frac{1}{3} + \frac{1}{6}\beta^+) = 2\int_0^1 A_U(\alpha)d\alpha.$$

Sometimes we are interested in the value and the ambiguity of fuzzy numbers. For a given reducing function $s = s(\alpha)$, by (1.74)-(1.75), these characteristics for a fuzzy number $W \in \mathbb{F}^p$ might be expressed as follows

$$Val_s(W) = \frac{w_1 + w_4}{2} + (w_2 - w_1)\int_0^1 s(\alpha)p(\alpha;\beta_0^-,\beta_1^-)d\alpha$$
$$+ (w_3 - w_4)\int_0^1 s(\alpha)p(\alpha;\beta_0^+,\beta_1^+)d\alpha$$

and

$$Amb_s(W) = \frac{w_4 - w_1}{2} + (w_1 - w_2)\int_0^1 s(\alpha)p(\alpha;\beta_0^-,\beta_1^-)d\alpha$$
$$+ (w_3 - w_4)\int_0^1 s(\alpha)p(\alpha;\beta_0^+,\beta_1^+)d\alpha.$$

The integrals above can be computed by numerical approximations (e.g. the trapezoidal or the Simpson formulas) but for specific cases, like $s(\alpha) = 1$ or $s(\alpha) = \alpha$, we can proceed analytically.

With reference to equations (3.246)-(3.247) let us define the following integrals depending on the slope parameters:

$$I(\beta_0, \beta_1) = \int_0^1 p(\alpha; \beta_0, \beta_1) d\alpha, \tag{3.250}$$

$$J(\beta_0, \beta_1) = \int_0^1 \alpha p(\alpha; \beta_0, \beta_1) d\alpha. \tag{3.251}$$

For the family of parametric functions $p_{MS}(\alpha; \beta_0, \beta_1)$ we obtain the nonlinear functions of β_0, β_1

$$I_{MS}(\beta_0, \beta_1) = \frac{2 + (3 + 2\beta_0)(\beta_0 + \beta_1)}{2(1 + \beta_0 + \beta_1)(2 + \beta_0 + \beta_1)},$$
$$J_{MS}(\beta_0, \beta_1) = \frac{42 + (84 + 40\beta_0)(\beta_0 + \beta_1)}{20(1 + \beta_0 + \beta_1)(2 + \beta_0 + \beta_1)(3 + \beta_0 + \beta_1)},$$

which become linear if, for example, we know or we assume a given value of $m = \beta_0 + \beta_1 \geq 0$. Note that fixing the value of m (we suggest m to be integer) is equivalent to fixing the "degree" of p_{MS} as a function of α. If m is integer then p_{MS} in a polynomial of degree one if $\beta_0 = \beta_1 = 1$, of degree two if $\beta_0 + \beta_1 = 2$, of degree three if $\beta_0 = \beta_1 = 0$ or of degree $m+1$ if $m = \beta_0 + \beta_1 \geq 3$. In these cases we obtain

$$I_{MS}(\beta_0, m - \beta_0) = \frac{2 + (3 + 2\beta_0)m}{2(1 + m)(2 + m)},$$
$$J_{MS}(\beta_0, m - \beta_0) = \frac{42 + (84 + 40\beta_0)m}{20(1 + m)(2 + m)(3 + m)},$$

with constraint $0 \leq \beta_0 \leq m$.

3.7.3 Nonlinear approximations with fixed support and core

Suppose we want to approximate given fuzzy number A by a fuzzy number $W \in \mathbb{F}^p$ which preserves the support and core of A. Assuming that $\mathrm{supp}(A) = [a_1, a_4]$ and $\mathrm{core}(A) = [a_2, a_3]$, our requirements immediately reduce to $w_i = a_i$ for $i = 1, \ldots 4$. Now we have to estimate the shape-parameters β_0^-, β_1^- and β_0^+, β_1^+ such that a distance measure $Dist(A, u)$ is minimized subject to the nonnegativity constraints on β_0^-, β_1^- and β_0^+, β_1^+. Here W results to be a function of $(\beta_0^-, \beta_1^-, \beta_0^+, \beta_1^+)$. So our problem is

$$Dist(A, W(\beta_0^-, \beta_1^-, \beta_0^+, \beta_1^+) \longrightarrow \min$$

with respect to $\beta_0^-, \beta_1^-, \beta_0^+, \beta_1^+ \geq 0$.

The distance $Dist(A, W(\beta_0^-, \beta_1^-, \beta_0^+, \beta_1^+))$ can be calculated if we have other information on A. For example, if the membership function of A is known at other points we can approximate the distance by the least squares functional. Suppose that $\mu_A(x_j) = \mu_j$ for given $x_j \in (a_1, a_2)$, $j = 1, 2, ..., j_L$ and for given $x_j \in (a_3, a_4)$, where $j = j_L + 1, j_L + 2, ..., j_L + j_R$ (i.e. j_L values correspond to the left side of the fuzzy number A while j_R values to the right side). So we should minimize

$$Dist(A, W(\beta_0^-, \beta_1^-, \beta_0^+, \beta_1^+)) = \tag{3.252}$$
$$= \sum_{j=1}^{j_L} \left\{ x_j - \left[a_1 + (a_2 - a_1) p(\mu_j; \beta_0^-, \beta_1^-) \right] \right\}^2$$
$$+ \sum_{j=j_L+1}^{j_L+j_R} \left\{ x_j - \left[a_4 + (a_3 - a_4) p(\mu_j; \beta_0^+, \beta_1^+) \right] \right\}^2$$

with respect to $\beta_0^-, \beta_1^-, \beta_0^+, \beta_1^+ \geq 0$.

Therefore, we have obtained a nonlinear least squares problem with four variables and nonnegativity constraints which can be solved by any numerical procedure. The minimization (3.252) can be split into two independent problems, one to determine β_0^-, β_1^- and the other for β_0^+, β_1^+.

A special case applies if we have available a single additional observation for the left side ($j_L = 1$), say $\mu_A(x^-) = \mu^-$ with $x^- \in (a_1, a_2)$, $\mu^- \in (0, 1)$ and for the right ($j_R = 1$), say $\mu_A(x^+) = \mu^+$ with $x^+ \in (a_3, a_4)$, $\mu^+ \in (0, 1)$.

An interpolating solution satisfies equations

$$a_1 + (a_2 - a_1) p(\mu^-; \beta_0^-, \beta_1^-) = x^-,$$
$$a_4 + (a_3 - a_4) p(\mu^+; \beta_0^+, \beta_1^+) = x^+.$$

If we use $p_{R2}(t; \beta_0, \beta_1)$ for $p(t; \beta_0, \beta_1)$, we obtain two equations and four nonnegative variables

$$(a_2 - x^-)\beta_0^- + (a_1 - x^-)\beta_1^- = \gamma^-, \tag{3.253}$$
$$(a_3 - x^+)\beta_0^+ + (a_4 - x^+)\beta_1^+ = \gamma^+, \tag{3.254}$$

where

$$\gamma^- = \frac{(a_1 - x^-)(2\mu^{-2} + 2\mu^- - 1) - (a_2 - a_1)\mu^{-2}}{\mu^-(1 - \mu^-)},$$
$$\gamma^+ = \frac{(a_4 - x^+)(2\mu^{+2} + 2\mu^+ - 1) - (a_3 - a_4)\mu^{+2}}{\mu^+(1 - \mu^+)}.$$

Equations (3.253)-(3.254) represent a line in the plane (β_0, β_1) having an infinite number of nonnegative solutions (note that $a_2 - x^- > 0$, $a_1 - x^- < 0$ and $a_3 - x^+ < 0$, $a_4 - x^+ > 0$). So we have many possible choices. We suggest three interesting solutions:

1. The unique solution minimizing $\beta_0^2 + \beta_1^2$ is obtained at the intersections of lines (3.253)-(3.254) with the axes and has the following closed form:

$$\begin{aligned} &\text{if } \gamma^- = 0 \quad \text{then } \beta_0^- = 0, \beta_1^- = 0; \\ &\text{if } \gamma^- > 0 \quad \text{then } \beta_0^- = \frac{\gamma^-}{b - x^-}, \beta_1^- = 0; \\ &\text{if } \gamma^- < 0 \quad \text{then } \beta_0^- = 0, \beta_1^- = \frac{\gamma^-}{a - x^-}. \end{aligned} \tag{3.255}$$

and

$$\begin{aligned} &\text{if } \gamma^+ = 0 \quad \text{then } \beta_0^+ = 0, \beta_1^+ = 0; \\ &\text{if } \gamma^+ > 0 \quad \text{then } \beta_0^+ = 0, \beta_1^+ = \frac{\gamma^+}{d - x^+}; \\ &\text{if } \gamma^+ < 0 \quad \text{then } \beta_0^+ = \frac{\gamma^+}{c - x^+}, \beta_1^+ = 0. \end{aligned} \tag{3.256}$$

Solutions (3.255)-(3.256) for two fuzzy numbers A_1 and A_2 with the same support and core given by $a_1 = 0$, $a_2 = 2.5$, $a_3 = 3$, $a_4 = 5$ and such that $\mu_{A_1}(0.5) = 0.5$, $\mu_{A_1}(3.5) = 0.9$ and $\mu_{A_2}(0.6) = 0.5$, $\mu_{A_2}(3.5) = 0.8$, are illustrated in Figure 3.10.

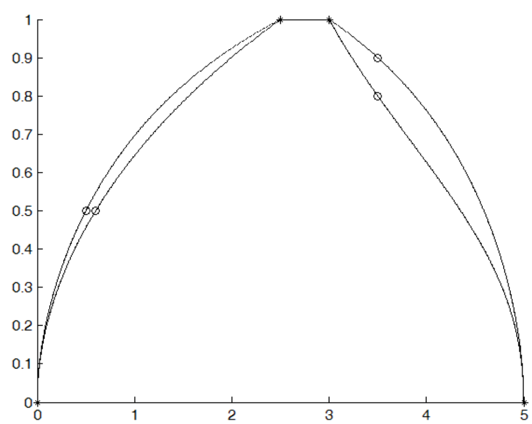

Fig. 3.10 Solutions (3.255)-(3.256) for two examples data.

2. Recall that functions $p(t;\beta_0,\beta_1)$ are linear if and only if $\beta_0 = \beta_1 = 1$. So it is reasonable to “measure” its nonlinearity by the distance between actual values of (β_0,β_1) and $(1,1)$. Moreover, one may be interested to find such (β_0,β_1) that minimize this distance, i.e. which minimize $(\beta_0 - 1)^2 + (\beta_1 - 1)^2$. The unique solution for this criterion is obtained by the following procedure:

$$\begin{aligned} &\text{if } \widehat{\beta}_0^- \geq 0 \text{ and } \widehat{\beta}_1^- \geq 0 \quad \text{then } \beta_0^- = \widehat{\beta}_0^-,\ \beta_1^- = \widehat{\beta}_1^-; \\ &\text{if } \widehat{\beta}_0^- < 0 \quad \text{then } \beta_0^- = 0,\ \beta_1^- = \frac{\gamma^-}{a_1 - x^-}; \\ &\text{if } \widehat{\beta}_1^- < 0 \quad \text{then } \beta_0^- = \frac{\gamma^-}{a_2 - x^-}, \beta_1^- = 0. \end{aligned} \tag{3.257}$$

where

$$\begin{aligned} \widehat{\beta}_0^- &= \frac{\gamma^-(a_2 - x^-) + (a_1 - x^-)^2 - (a_2 - x^-)(a_1 - x^-)}{(a_2 - x^-)^2 + (a_1 - x^-)^2} \\ \widehat{\beta}_1^- &= 1 + \frac{(a_1 - x^-)(\widehat{\beta}_0^- - 1)}{(a_2 - x^-)}; \end{aligned}$$

for β_0^+, β_1^+ the procedure is analogous

$$\begin{aligned} &\text{if } \widehat{\beta}_0^+ \geq 0 \text{ and } \widehat{\beta}_1^+ \geq 0 \quad \text{then } \beta_0^+ = \widehat{\beta}_0^+,\ \beta_1^+ = \widehat{\beta}_1^+; \\ &\text{if } \widehat{\beta}_0^+ < 0 \quad \text{then } \beta_0^+ = 0,\ \beta_1^+ = \frac{\gamma^+}{a_3 - x^+}; \\ &\text{if } \widehat{\beta}_1^+ < 0 \quad \text{then } \beta_0^+ = \frac{\gamma^+}{a_4 - x^+}, \beta_1^+ = 0. \end{aligned} \tag{3.258}$$

where

$$\begin{aligned} \widehat{\beta}_0^+ &= \frac{\gamma^+(a_3 - x^+) + (a_4 - x^+)^2 - (a_4 - x^+)(a_3 - x^+)}{(a_3 - x^+)^2 + (a_4 - x^+)^2} \\ \widehat{\beta}_1^+ &= 1 + \frac{(a_4 - x^+)(\widehat{\beta}_0^+ - 1)}{(a_3 - x^+)}. \end{aligned}$$

and

$$\begin{aligned} &\text{if } \gamma^+ = 0 \quad \text{then } \beta_0^+ = 0, \beta_1^+ = 0; \\ &\text{if } \gamma^+ > 0 \quad \text{then } \beta_0^+ = \frac{\gamma^+}{a_4 - x^+}, \beta_1^+ = 0; \\ &\text{if } \gamma^+ < 0 \quad \text{then } \beta_0^+ = 0, \beta_1^+ = \frac{\gamma^+}{a_3 - x^+}. \end{aligned} \tag{3.259}$$

Solutions (3.257)-(3.258) for the same data as before are illustrated in Figure 3.11.

3. In cases 1. and 2. we made no assumptions on the parameters β_0^-, β_1^-, β_0^+, β_1^+ but in practise we may know or desire to have them partially fixed. For example, we require a differentiable membership function and this is equivalent to have $\beta_{1,L} = 0$ and $\beta_{1,R} = 0$. Writing the fitting equations in terms of the *L-R* representation (3.245) we obtain the following solution

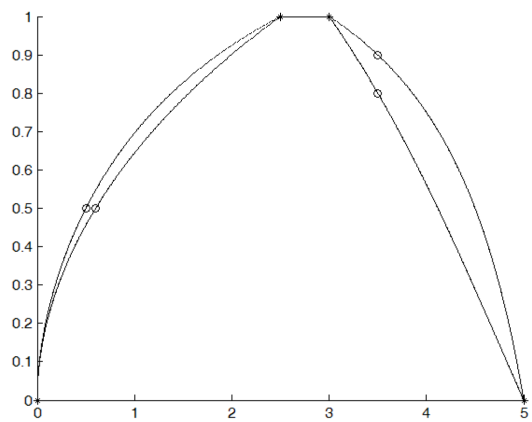

Fig. 3.11 Solutions (3.257)-(3.258) for two examples data.

$$\beta_{0,L} = \begin{cases} \frac{\omega^-}{t^-(1-t^-)(1-\mu^-)} & \text{if } \omega^- > 0 \\ 0 & \text{if } \omega^- \leq 0, \end{cases} \tag{3.260}$$

where $t^- = \frac{x^- - a}{b-a}$ and $\omega^- = (1-2t^-)\mu^- - (1-2\mu^-)(t^-)^2$,

$$\beta_{0,R} = \begin{cases} \frac{\omega^+}{t^+(1-t^+)\mu^+} & \text{if } \omega^+ > 0 \\ 0 & \text{if } \omega^+ \leq 0, \end{cases} \tag{3.261}$$

where $t^+ = \frac{x^+ - a_3}{a_4 - a_3}$ and $\omega^+ = (1-\mu^+)(1-2t^+) - (2\mu^+ - 1)(t^+)^2$. Solution (3.260)-(3.261) for the same data as considered before are illustrated in Figure 3.12.

3.7.4 Nonlinear approximations with fixed shapes

Suppose now that the general shape of W is fixed by determining values of the parameters $(\beta_0^-, \beta_1^-, \beta_0^+, \beta_1^+))$ and we like to find such W which approximates best the core and support of A, i.e. to find w_1, w_2, w_3 and w_4 such that

$$Dist(A, W(w_1, w_2, w_3, w_4)) \to \min.$$

Actually, now W is a function of w_1, w_2, w_3, w_4 that should fulfill the requirement $w_1 \leq w_2 \leq w_3 \leq w_4$.

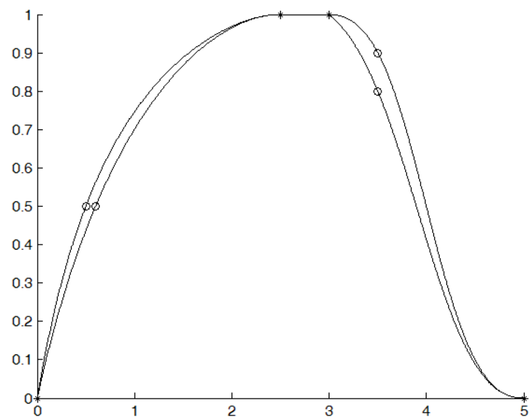

Fig. 3.12 Solutions (3.260)-(3.261) for two examples data.

In this case the resulting optimization problem is easier than in Section 3.7.3. Suppose, as before, that we know the membership of A at $j_L + j_R$ points, $\mu_A(x_j) = \mu_j$ for given $x_j \in (a_1, a_2)$, $j = 1, 2, ..., j_L$ and for given $x_j \in (a_3, a_4)$, $j = j_L + 1, j_L + 2, ..., j_L + j_R$. Define $p_j = p(\mu_j; \beta_0^-, \beta_1^-)$ for $j = 1, 2, ..., j_L$ and $p_j = p(\mu_j; \beta_0^+, \beta_1^+)$ for $j = j_L + 1, j_L + 2, ..., j_L + j_R$.

Now we have to minimize

$$Dist(A, W(w_1, w_2, w_3, w_4)) = \sum_{j=1}^{j_L} \left\{x_j - [w_1 + (w_2 - w_1)p_j]\right\}^2 + \sum_{j=j_L+1}^{j_L+j_R} \left\{x_j - [w_3 + (w_3 - w_4)p_j]\right\}^2 \tag{3.262}$$

where

$$w_1 - w_2 \leq 0$$
$$w_2 - w_3 \leq 0$$
$$w_3 - w_4 \leq 0.$$

We obtain a linear least squares problem with four variables and three linear constraints that can be solved using standard well known procedures (see, e.g., [143]).

3.8 Fuzzy number approximation via shadowed sets

3.8.1 Shadowed sets

A membership function indicates a grade to which a given point in the universe of discourse belongs to a concept under study described by given fuzzy set. Numerous methods for constructing membership functions have been described in the literature (see, e.g., [112] or [137] and references given there) and it has been found that the most uncertainty in the determination of the membership function is associated with those grades situated around 0.5. In contrast, one is usually much more confident in assigning values close to 1 or close to 0 corresponding to elements which might be surely included or excluded from the concept, respectively. This observation led Pedrycz to the idea of shadowed sets ([163] and developed later [165, 164]) which form an alternative way of modeling vagueness that relies on basic concepts of truth values (yes/no) and on entire unit interval perceived as a zone of uncertainty.

Although one can model an imprecise object under study directly by an appropriate shadowed set, these very shadowed sets might be also conceived as another method for simplifying fuzzy sets. More precisely, starting with the initial fuzzy set we may try to construct a corresponding shadowed set that capture the essence of the fuzzy set while reducing simultaneously computational efforts and simplifying the interpretation. This perspective was suggested from the very beginning by Pedrycz who even proposed an approximation algorithm (see [163]). Another algorithm for fuzzy number approximation that throws a bridge to interval and trapezoidal approximation mentioned above was proposed by Grzegorzewski [114]. However, before showing it let us define formally shadowed sets.

Formally speaking, a **shadowed set** S in a universe of discourse $\mathbb{X}$ is a set-valued mapping $S : \mathbb{X} \to \{0, [0,1], 1\}$ having the following interpretation (see [163]):

- all elements of $\mathbb{X}$ for which $S(x) = 1$ are called a core of the shadowed set S and they embraced all elements that are fully compatible with the concept conveyed by S,
- all elements of $\mathbb{X}$ for which $S(x) = 0$ are completely excluded from the concept described by S,
- all elements of $\mathbb{X}$ for which $S(x) = [0,1]$, called a *shadow*, are uncertain.

Thus for a shadowed set S we define the core as $\mathrm{core}(S) = \{x \in \mathbb{X} : S(x) = 1\}$, the shadow by $\mathrm{sh}(S) = \{x \in \mathbb{X} : S(x) = [0,1]\}$ and the support by $\mathrm{supp}(S) = cl(\{x \in \mathbb{X} : S(x) \neq 0\})$.

The usage of the unit interval in this last case reflects and helps quantify the effect of hesitation and shows that any element from the shadow could be excluded or exhibit partial membership or could be fully allocated to S.

One can view shadowed sets as an example of a three-valued logic. In particular, basic operations on shadowed sets are isomorphic with those encountered in the three-valued logic, i.e.

- $A \cup B$

$A \setminus B$	0	$[0,1]$	1
0	0	$[0,1]$	1
$[0,1]$	$[0,1]$	$[0,1]$	1
1	1	1	1

- $A \cap B$

$A \setminus B$	0	$[0,1]$	1
0	0	0	0
$[0,1]$	0	$[0,1]$	$[0,1]$
1	0	$[0,1]$	1

- $\neg A$

A	$\neg A$
0	1
$[0,1]$	$[0,1]$
1	0

Further on we consider shadowed sets defined on the real line, i.e. when $\mathbb{X} = \mathbb{R}$. An example of such shadowed set is given in Figure 3.13, where $\text{core}(S) = [s_2, s_3] = [2,4]$, $\text{sh}(S) = (s_1, s_2) \cup (s_3, s_4) = [1,2] \cup [4,5]$ and $\text{supp}(S) = [s_1, s_4] = [1,5]$.

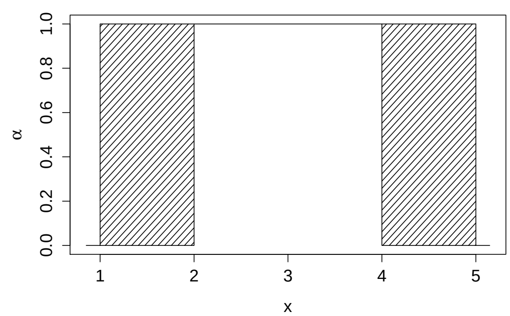

Fig. 3.13 An example of a shadowed set.

3.8.2 The shadowed set approximation

Our aim now is to construct a shadowed set preserving the uncertainty associated with the original fuzzy set. When trying to achieve this goal we have to keep in mind that while in fuzzy set we encounter intermediate membership grades located between 0 and 1, in shadowed set we have to allocate the entire uncertainty in a compact shadow. We create this shadow using two successive interval approximations (see Section 3.3). This idea comes from the observation that a shadowed set A might be perceived as a conjunction of two rectangular fuzzy numbers: the "wider" one describing all points which possibly belong to A and the second one - more "narrow" - corresponding to elements almost surely belonging to A. Since degrees of membership both high (close to 1) and low (close to 0) are much more informative than those close to 0.5 we try to combine two interval approximations for two different weighting functions: the first one with increasing weighting function $w^*(\alpha)$ and the second one with decreasing weighting function $w^{**}(\alpha)$. Therefore, our goal is to reach two intervals $C^*(A) = [C_L^*, C_U^*]$ and $C^{**}(A) = [C_L^{**}, C_U^{**}]$ that correspond to our optimization problem under $w^*(\alpha)$ and $w^{**}(\alpha)$, respectively. In this section we consider two following weighting functions: $w^*(\alpha) = \alpha$ and $w^{**}(\alpha) = 1 - \alpha$.

Thus, by (1.42), we have to minimize

$$d_w^2(A, C^*(A)) = \int_0^1 \alpha (A_L(\alpha) - C_L^*)^2 d\alpha + \int_0^1 \alpha (A_U(\alpha) - C_U^*)^2 d\alpha \tag{3.263}$$

with respect to C_L^* and C_U^* and to minimize

$$d_w^2(A, C^{**}(A)) = \int_0^1 (1-\alpha)(A_L(\alpha) - C_L^{**})^2 d\alpha + \int_0^1 (1-\alpha)(A_U(\alpha) - C_U^{**})^2 d\alpha \tag{3.264}$$

with respect to C_L^{**} and C_U^{**}.

In order to minimize $D(C_L^*, C_U^*) = d_w^2(A, C^*(A))$ we have to solve a following system of equations

$$\begin{cases} \frac{\partial D(C_L^*, C_U^*)}{\partial C_L^*} = 0 \\ \frac{\partial D(C_L^*, C_U^*)}{\partial C_U^*} = 0 \end{cases}$$

which is equivalent to

$$\begin{cases} \int_0^1 \alpha (A_L(\alpha) - C_L^*) d\alpha = 0 \\ \int_0^1 \alpha (A_U(\alpha) - C_U^*) d\alpha = 0. \end{cases}$$

The solution is

$$C_L^* = 2 \int_0^1 \alpha A_L(\alpha) d\alpha \tag{3.265}$$

$$C_U^* = 2 \int_0^1 \alpha A_U(\alpha) d\alpha. \tag{3.266}$$

After simple calculations it is easy to prove that $C^*(A) = [C_L^*, C_U^*]$ with the borders (3.265) and (3.266) is indeed the nearest interval approximation of fuzzy number A with respect to metric (3.263). Similarly, when minimizing function $D(C_L^{**}, C_U^{**}) = d_w^2(A, C^{**}(A))$ we obtain $C^{**}(A) = [C_L^{**}, C_U^{**}]$ with the borders given by

$$C_L^{**} = 2\int_0^1 (1-\alpha)A_L(\alpha)d\alpha = 2\int_0^1 A_L(\alpha)d\alpha - C_L^*, \tag{3.267}$$

$$C_U^{**} = 2\int_0^1 (1-\alpha)A_U(\alpha)d\alpha = 2\int_0^1 A_U(\alpha)d\alpha - C_U^*. \tag{3.268}$$

It can be shown that

Lemma 3.8. *For any fuzzy number A we have*

$$C^*(A) \subseteq C^{**}(A). \tag{3.269}$$

Proof. We have to show that $C_L^{**} \leq C_L^*$and $C_U^* \leq C_U^{**}$. Actually,

$$\begin{aligned}
C_L^* - C_L^{**} &= 4\int_0^1 \alpha A_L(\alpha)d\alpha - 2\int_0^1 A_L(\alpha)d\alpha \\
&= 2\int_0^1 (2\alpha-1)A_L(\alpha)d\alpha \\
&= 2\left[\int_0^{1/2}(2\alpha-1)A_L(\alpha)d\alpha + \int_{1/2}^1 (2\alpha-1)A_L(\alpha)d\alpha\right] \\
&= 2\left[\int_0^{1/2}(2\alpha-1)A_L(\alpha)d\alpha - \int_0^{1/2}(2\gamma-1)A_L(1-\gamma)d\gamma\right] \\
&= 2\int_0^{1/2}(2\alpha-1)[A_L(\alpha)-A_L(1-\alpha)]d\alpha.
\end{aligned}$$

Since $0 \leq \alpha \leq \frac{1}{2}$then $2\alpha - 1 \leq 0$ and $A_L(\alpha) - A_L(1-\alpha) \leq 0$. Consequently $C_L^* - C_L^{**} \geq 0$. Similarly we can show that $C_U^* - C_U^{**} \leq 0$ which proves the lemma. ■

Now we are able to present our shadowed set approximation operator. Let $\mathbb{SH}(\mathbb{R})$ denote a family of shadowed sets on $\mathbb{R}$.

Definition 3.16. A shadowed set approximation operator nearest to given fuzzy number A is an operator $S : \mathbb{F}(\mathbb{R}) \to \mathbb{SH}(\mathbb{R})$ that produces a shadowed set $S(A)$ given by points $S(A) = (s_1, s_2, s_3, s_4) = (C_L^{**}, C_L^*, C_U^*, C_U^{**})$, where

$$C_L^{**} = 2\int_0^1 A_L(\alpha)d\alpha - 2\int_0^1 \alpha A_L(\alpha)d\alpha, \tag{3.270}$$

$$C_L^* = 2\int_0^1 \alpha A_L(\alpha)d\alpha, \tag{3.271}$$

$$C_U^* = 2\int_0^1 \alpha A_U(\alpha)d\alpha, \tag{3.272}$$

$$C_U^{**} = 2\int_0^1 A_U(\alpha)d\alpha - 2\int_0^1 \alpha A_U(\alpha)d\alpha. \tag{3.273}$$

Going back our previous remarks and formulae we can conclude that

$$\mathrm{core}(S(A)) = [C_L^*, C_U^*],$$

while the shadow is given by

$$\mathrm{sh}(S(A)) = (C_L^{**}, C_L^*) \cup (C_U^*, C_U^{**})$$

and the support is

$$\mathrm{supp}(S(A)) = [C_L^{**}, C_U^{**}].$$

One may notice that the points describing the shadowed-set approximation of a fuzzy number might be expressed by some characteristics of that fuzzy number. Actually, by (1.78) and (1.79) we get immediately

$$Val(A) + Amb(A) = 2\int_0^1 \alpha A_U(\alpha) d\alpha,$$
$$Val(A) - Amb(A) = 2\int_0^1 \alpha A_L(\alpha) d\alpha.$$

Hence by (1.71) we get the shadowed set representation equivalent to that given in Definition 3.16. Namely

Lemma 3.9. *The shadowed set approximation operator nearest to given fuzzy number A is an operator $S : \mathbb{F}(\mathbb{R}) \to \mathbb{SH}(\mathbb{R})$ that produces a shadowed set $S(A)$ given by points $S(A) = (s_1, s_2, s_3, s_4) = (C_L^{**}, C_L^*, C_U^*, C_U^{**})$, where*

$$C_L^{**} = 2EI_L(A) - C_L^*, \tag{3.274}$$
$$C_L^* = Val(A) - Amb(A), \tag{3.275}$$
$$C_U^* = Val(A) + Amb(A), \tag{3.276}$$
$$C_U^{**} = 2EI_U(A) - C_U^*. \tag{3.277}$$

Let us consider the following example.

Example 3.10. Suppose a fuzzy number A has a following membership function (see Example 3.2)

$$\mu_A(x) = \begin{cases} 1 - x^2 & \text{if} \quad -1 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

The α-cuts of A are of the form $A_\alpha = [-\sqrt{1-\alpha}, \sqrt{1-\alpha}]$. Using our approximation method we get a following shadowed set $S(A) = (-\frac{8}{10}, -\frac{8}{15}, \frac{8}{15}, \frac{8}{10})$. The original fuzzy set and its shadowed set approximation are shown in Figure 3.14.

As it was noticed in [120] there is one-to-one correspondence between shadowed sets and the particular families of intuitionistic fuzzy sets or interval-valued fuzzy sets (see Chapter 2). Actually, from the mathematical point of view, a shadow set is in fact nothing else than as an intuitionistic fuzzy set such that its membership function μ and nonmembership function ν are of the type $\mu, \nu : \mathbb{X} \to \{0, 1\}$. Or,

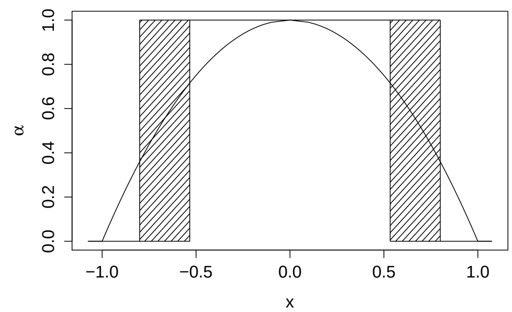

Fig. 3.14 Fuzzy number approximation by a shadowed set, see Example 3.10.

using the second mentioned formalism, a shadow set is such interval-valued fuzzy set where a lower fuzzy set and a upper fuzzy sets of take values not in the interval $[0,1]$ but in $\{0,1\}$. However, shadow sets have quite nice and natural interpretation and thus there is nothing wrong in that they exist under their own name. Anyway, we may utilize concepts that appear in the theory of intuitionistic fuzzy sets (interval-valued fuzzy sets) to get the corresponding definitions for shadowed sets.

Let us turn our attention for a moment to intuitionistic fuzzy sets for some tools required for further considerations. It is worth noting that each intuitionistic fuzzy number $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle$ is a conjunction of two fuzzy numbers: A^+ with a membership function $\mu_{A^+}(x) = \mu_A(x)$ and A^- with a membership function $\mu_{A^-}(x) = 1 - \nu_A(x)$. It is seen that $\text{supp}A^+ \subseteq \text{supp}A^-$. Moreover, we have $EI(A^+) \subseteq EI(A^-)$, where $EI(A^+) = [EI_L(A^+), EI_U(A^+)]$ and $EI(A^-) = [EI_L(A^-), EI_U(A^-)]$. Hence we can define a expected interval of an intuitionistic fuzzy number $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle$ as a crisp interval $\widetilde{EI}(A_{\langle\rangle})$ given by (see [106])

$$\widetilde{EI}(A_{\langle\rangle}) = \left[\widetilde{EI_L}(A_{\langle\rangle}), \widetilde{EI_U}(A_{\langle\rangle})\right],$$

where

$$\widetilde{EI_L}(A_{\langle\rangle}) = \frac{EI_L(A^-) + EI_L(A^+)}{2},$$
$$\widetilde{EI_U}(A_{\langle\rangle}) = \frac{EI_U(A^-) + EI_U(A^+)}{2}.$$

In the same way we may generalize the notion of the width, given by (1.82), into intuitionistic fuzzy numbers. Thus width of an intuitionistic fuzzy number $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle$ is a real number given by

$$\widetilde{w}(A_{\langle\rangle}) = \frac{1}{2}\left[\int_{-\infty}^{\infty} \mu_A(x)dx + \int_{-\infty}^{\infty} (1 - \nu_A(x))dx\right].$$

One may easily seen that

$$\widetilde{w}(A_{\langle\rangle}) = \frac{w(A^+) + w(A^-)}{2}.$$

It can be shown, that similarly as in the case of fuzzy numbers, the width of an intuitionistic fuzzy number is equal to the length of the expected interval corresponding to this intuitionistic fuzzy number, i.e.

$$\widetilde{w}(A_{\langle\rangle}) = \widetilde{EI_U}(A_{\langle\rangle}) - \widetilde{EI_L}(A_{\langle\rangle}).$$

Now, using the notions given above for a shadowed set $S = S(s_1, s_2, s_3, s_4)$ we get immediately

$$\widetilde{EI}(S) = \left[\frac{s_1 + s_2}{2}, \frac{s_3 + s_4}{2}\right]$$

and

$$\widetilde{w}(S) = \frac{s_3 + s_4}{2} - \frac{s_1 + s_2}{2}.$$

Now, after introducing these concepts, we may discuss some interesting properties of the shadowed set approximation. As we have shown in Section 3.3 the nearest interval approximation operator preserves the expected interval of a fuzzy number. As an immediate consequence, this operator preserves the expected value and the width of fuzzy numbers. It appears that this nice and desired property also holds for shadowed set approximation i.e.

$$\widetilde{EI}(S(A)) = EI(A)$$

and

$$\widetilde{w}(S(A)) = w(A).$$

For easy proofs we refer the reader to [114].

Problems

3.1. Let $T : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^T(\mathbb{R})$ be the trapezoidal approximation operator preserving the expected interval proposed in Subsection 3.5.1. Compute $T(A)$ if

a) $A_L(\alpha) = 1 + \sqrt{\alpha}$ and $A_U(\alpha) = 30 - 27\sqrt{\alpha}$;
b) $A_L(\alpha) = 2\alpha - 2$ and $A_U(\alpha) = 1 - \sqrt{\alpha}$.

3.2. Let $T : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^T(\mathbb{R})$ be the trapezoidal approximation operator preserving the ambiguity and value proposed in Subsection 3.5.4. Compute $T(A)$ if

a) $A_L(\alpha) = 1 + \alpha^2$ and $A_U = 4 - \alpha$;
b) $A_L(\alpha) = 2\alpha - 20$ and $A_U(\alpha) = 1 - \sqrt{\alpha}$.

3.3. A distance d on $\mathbb{F}(\mathbb{R})$ is called translation invariant if $d(A+C,B+C)=d(A,B)$, for all $A,B,C\in\mathbb{F}(\mathbb{R})$. Let $P_k:\mathbb{F}(\mathbb{R})\to\mathbb{R}$, where $k=1,\ldots,n$, be parameters associated with fuzzy numbers such that

$$P_k(A+z)=P_k(A)+f_k(z),$$

for every $A\in\mathbb{F}(\mathbb{R})$ and $z\in\mathbb{R}$, where, f_k, $k=1,\ldots,n$, are real functions of real variable. If $\Omega\subset\mathbb{F}(\mathbb{R})$ satisfies $z+\Omega=\Omega\ \forall z\in\mathbb{R}$ and $\omega(A)\in\Omega$ is the nearest fuzzy number to a given $A\in\mathbb{F}(\mathbb{R})$ (with respect to d) which preserves P_k for $k\in\{1,\ldots,n\}$, i.e.

$$P_k(\omega(A))=P_k(A),\quad \forall k\in\{1,\ldots,n\},$$

then $\omega(A)+z\in\Omega$ is the nearest fuzzy number to $A+z$ (with respect to D) which preserves P_k for $k\in\{1,\ldots,n\}$, i.e.

$$P_k(\omega(A)+z)=P_k(A+z),\quad \forall k\in\{1,\ldots,n\}.$$

3.4. Let $T:\mathbb{F}(\mathbb{R})\to\mathbb{F}^T(\mathbb{R})$ be either the trapezoidal approximation operator preserving the expected interval or the trapezoidal approximation operator preserving the value and the ambiguity. Prove that $T(A+z)=T(A)+z$, for all $A\in\mathbb{F}(\mathbb{R})$ and $z\in\mathbb{R}$.

3.5. A distance d on $\mathbb{F}(\mathbb{R})$ is called homogeneous if $d(\lambda A,\lambda B)=|\lambda|\,d(A,B)$, for all $A,B\in\mathbb{F}(\mathbb{R})$ and $\lambda\in\mathbb{R}$. Let $P_k:\mathbb{F}(\mathbb{R})\to\mathbb{R}$, where $k=1,\ldots,n$, be parameters associated with fuzzy numbers such that

$$P_k(\lambda\cdot A)=\lambda P_k(A),$$

for every $A\in\mathbb{F}(\mathbb{R})$ and $\lambda\in\mathbb{R}$ or

$$P_k(\lambda\cdot A)=|\lambda|\,P_k(A),$$

for every $A\in\mathbb{F}(\mathbb{R})$ and $\lambda\in\mathbb{R}$. If $\Omega\subset\mathbb{F}(\mathbb{R}),\lambda\cdot\Omega\subset\Omega,\forall\lambda\in\mathbb{R}$ and $\omega(A)\in\Omega$ is the nearest fuzzy number to a given $A\in\mathbb{F}(\mathbb{R})$ (with respect to D) which preserves P_k for $k\in\{1,\ldots,n\}$, i.e.

$$P_k(\omega(A))=P_k(A),\quad \forall k\in\{1,\ldots,n\},$$

then $\lambda\cdot\omega(A)\in\Omega$ is the nearest fuzzy number to $\lambda\cdot A$ (with respect to D) which preserves P_k for $k\in\{1,\ldots,n\}$, i.e.

$$P_k(\omega(\lambda\cdot A))=P_k(\lambda\cdot A),\quad \forall k\in\{1,\ldots,n\}.$$

3.6. Let $T:\mathbb{F}(\mathbb{R})\to\mathbb{F}^T(\mathbb{R})$ be either the trapezoidal approximation operator preserving the expected interval or the trapezoidal approximation operator preserving the value and the ambiguity. Prove that $T(\lambda A)=\lambda T(A)$, for all $A\in\mathbb{F}(\mathbb{R})$ and $\lambda\in\mathbb{R}$.

3.7. We call a fuzzy number A symmetric if the function A_L+A_U is constant. Let $T:\mathbb{F}(\mathbb{R})\to\mathbb{F}^T(\mathbb{R})$ be either the trapezoidal approximation operator preserving the expected interval or the trapezoidal approximation operator preserving the value and the ambiguity. Prove that if A is symmetric then $T(A)$ is symmetric too.

3.8. Let us consider the trapezoidal valued operator $T : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^T(\mathbb{R})$ which satisfies the condition that if $A \in \mathbb{F}(\mathbb{R})$ satisfies $T_e(A) \in \mathbb{F}^T(\mathbb{R})$, then $T(A) = T_e(A)$. Prove that T is non-additive. Here, $T_e(A)$ is the extended trapezoidal approximation of A (see (3.179)-(3.182)).

3.9. Let $T : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^T(\mathbb{R})$ be either the trapezoidal approximation operator preserving the expected interval or the trapezoidal approximation operator preserving the value and the ambiguity. Prove that T is non-additive.

Chapter 4
Ranking fuzzy numbers

4.1 Introduction to the topic

In the last decades many papers were devoted to studies on fuzzy number ranking procedures. We can distinguish two main approaches.

The first one is based on so called ranking indices. They are functions from fuzzy numbers to real values and a ranking is generated by a procedure based on the standard ordering of reals. It is the most often used idea (see, e.g., [3, 4, 6, 7, 11, 12, 56, 57, 67, 69, 84, 88, 135, 148, 154, 171, 184, 192]) and it was extended for other classes of fuzzy sets which are not necessarily fuzzy numbers (see, e.g., [58, 141]).

The second one is based on fuzzy binary relations (see, e.g., [15, 80, 86, 157]). This approach can be also extended to more general settings than fuzzy numbers (see [9]).

As a conclusion, we see that there are numerous ways to rank fuzzy numbers. Some comparative studies can be found in [47, 49] or [184].

The main results included in the present chapter are helpful for studying which reasonable properties are satisfied by some ranking procedure. In addition, the results describe precisely the shape of a ranking index which generates an order that satisfies a specific requirement. The impact of a good choice of the ranking of fuzzy numbers is decisive in applications related with decision theory, optimization, artificial intelligence, approximate reasoning, socioeconomic systems and so on. In fact, a key issue in operationalizing fuzzy set theory is how to compare fuzzy numbers (see [151]).

In this chapter we discuss only ranking approaches obtained from ranking indices. In the recent papers (see, e.g., [6, 11, 12, 88, 141, 171, 172, 187]) the authors try to impose a certain ranking method by finding some examples in which their approach gives better results comparing to others. As it is pointed out in [36] there is a kind of subjectivity because from a few examples we cannot conclude which approach is better. This is why our aim is to characterize ranking approaches rather than to classify them, starting from the reasonable properties of Wang and Kerre [184] presented in Section 4.2. Because the set of trapezoidal fuzzy numbers and the set of triangular fuzzy numbers are closed under addition and scalar multiplication, we are mostly interested in such sets of fuzzy numbers. As an immediate consequence of the fact that they are not closed under fuzzy multiplication we consider a slight modification of the requirement “If $A \succeq B$ then $A \cdot C \succeq B \cdot C$ for every $C \geq 0$” in [184]. Actually, except the approach in [7], we are not aware of the existence of an ordering generated by a ranking index which would satisfy the above property.

This chapter is organized as follows. In Section 4.2 we discuss on the basic requirements for the case when the ordering on a set of fuzzy numbers is induced by a binary relation. The main result in Section 4.2 asserts that, making abstraction of equivalent orders, one needs to discuss only ranking indices with the property that they belong to the support of fuzzy number. The study of the ranking indices is drastically simplified, as we can see in Section 4.3. Actually, if a ranking index is defined on a closed under additive and scalar multiplication set of fuzzy numbers and it generates an order satisfying all the reasonable properties then there exists a linear ranking index which generates an equivalent order. In Sections 4.4 and 4.5 we take advantage of the results obtained in the previous sections for the ranking of triangular fuzzy numbers and trapezoidal fuzzy numbers, respectively. We conclude that making abstraction of equivalent orders we can determine exactly the class of ranking indices which generate orders satisfying all the basic requirements. In addition, we observe that we can characterize classes of ranking indices generating orders which satisfy just a part of the reasonable properties. The benefits of these results are obvious taking into account that in most applications the researchers use triangular or trapezoidal fuzzy numbers (see, e.g., [23, 59, 150, 183]). In Section 4.6 we prove how the orders between trapezoidal fuzzy numbers can be extended to orders between arbitrary fuzzy numbers such that some (or all) of the basic desirable properties are preserved.

Even if the chapter is essentially based on the results in [35, 36] and [73], the results are more complete and the approach more rigorous.

4.2 Reasonable properties for ranking fuzzy numbers

The quality of an ordering approach over a set $\mathscr{S} \subseteq \mathbb{F}(\mathbb{R})$ is decisive in applications. It is commonly agreed that the list of reasonable properties proposed by Wang and Kerre [184] is the most objective way in validating a certain ranking approach. We write the axioms for the particular case when $\succeq$ is a binary relation over $\mathscr{S}$. It is worth mentioning that Wang and Kerre considered a more general approach starting from so called ranking indices of the second kind. Moreover, we suppose that $\succeq$ is a total order on $\mathscr{S}$ which means that for any $(A,B) \in \mathscr{S}^2$ either $A \succeq B$ or $B \succeq A$. Then we can easily construct other binary relations derived from $\succeq$. At first, we construct on $\mathscr{S}$ the relation $\not\succeq$ which is the negation of $\succeq$, that is $A \not\succeq B \Leftrightarrow (A,B) \notin \mathscr{S}_{\succeq}$, where $\mathscr{S}_{\succeq} = \{(A,B) \in \mathscr{S}^2 : A \succeq B\}$. Next we construct the relation $\succ$ on $\mathscr{S}$, where $A \succ B \Leftrightarrow B \not\succeq A$. Obviously this implies that from $A \succ B$ it results that $A \succeq B$. If not, then it easily results that $A \not\succeq B$ and $B \not\succeq A$ which contradicts the fact that $\succeq$ is a total binary relation on $\mathscr{S}$. Note that if $\succeq$ would not be a total binary relation then the correct definition of $\succ$ would be $A \succ B \Leftrightarrow A \succeq B$ and $B \not\succeq A$. Furthermore, we construct the relation $\sim$ on $\mathscr{S}$, where $A \sim B \Leftrightarrow A \succeq B$ and $B \succeq A$. It is immediate that $\sim$ is an equivalence relation over $\mathscr{S}$. Also of note is that $A \succeq B \Leftrightarrow A \succ B$ or $A \sim B$ for all $A, B \in \mathscr{S}$. Similarly, we can define on $\mathscr{S}$ the relations $\preceq$, $\prec$, $\not\preceq$ and $\nsim$ respectively. We do not go into details since their construction is obvious.

For the binary relations from above we consider the following basic requirements:

$\mathbb{A}_1$) $A \succeq A$ for any $A \in \mathscr{S}$.
$\mathbb{A}_2$) For any $(A, B) \in \mathscr{S}^2$, if $A \succeq B$ and $B \succeq A$ then $A \sim B$.
$\mathbb{A}_3$) For any $(A, B, C) \in \mathscr{S}^3$, if $A \succeq B$ and $B \succeq C$ then $A \succeq C$.
$\mathbb{A}_4$) For any $(A, B) \in \mathscr{S}^2$, if $\inf \mathrm{supp}(A) \geq \sup \mathrm{supp}(B)$ then $A \succeq B$.
$\mathbb{A}'_4$) For any $(A, B) \in \mathscr{S}^2$, if $\inf \mathrm{supp}(A) > \sup \mathrm{supp}(B)$ then $A \succ B$.
$\mathbb{A}_5$) Let $A, B, A + C$ and $B + C$ be elements of $\mathscr{S}$. If $A \succeq B$, then $A + C \succeq B + C$.
$\mathbb{A}'_5$) Let $A, B, A + C$ and $B + C$ be elements of $\mathscr{S}$. If $A \succ B$, then $A + C \succ B + C$.
$\mathbb{A}_6$) For any $(A, B) \in \mathscr{S}^2$ and $\lambda \in \mathbb{R}$ such that $\lambda \cdot A, \lambda \cdot B \in \mathscr{S}$, if $A \succeq B$ then $\lambda \cdot A \succeq \lambda \cdot B$ for $\lambda \geq 0$ and $\lambda \cdot A \preceq \lambda \cdot B$ for $\lambda \leq 0$.

If $\mathbb{A}_1$ and $\mathbb{A}_3$ are satisfied then $\succeq$ is a total preorder on $\mathscr{S}$ which in Order Theory stands as minimal requirement that a binary relation should satisfy (see also our paper [36] for more details).

The above list of requirements is a little bit different to what we can find in the paper of Wang and Kerre (for more details we refer again to paper [36]). We just mention that the axiom denoted with $\mathbb{A}_5$ in [184] is eliminated from the above list because it holds trivially in the case of binary relations. We also have to notice that the axiom $\mathbb{A}_6$ from our list has in some sense a more restrictive form in paper [184] but this stronger form has a limitation because it cannot be applied on sets which are not closed under multiplication such as the set of trapezoidal or triangular fuzzy numbers. Definitely $\mathbb{A}_6$ in the present form has its importance since for $A \succeq B$ it results in $-B \succeq -A$, a quite natural property considered important in many papers (see, e.g., [4, 11, 88]).

Another important issue is how to generate an effective binary relation $\succeq$. Most often, so called ranking index (also known as utility function in economics), that is a function $P : \mathscr{S} \to \mathbb{R}$, is used. This function gives a binary relation $\succeq_P$, where $A \succeq_P B$ if and only if $P(A) \geq P(B)$. This easily implies that

$$\begin{aligned} A \preceq_P B &\quad \text{if and only if} \quad P(A) \leq P(B), \\ A \succ_P B &\quad \text{if and only if} \quad P(A) > P(B), \\ A \prec_P B &\quad \text{if and only if} \quad P(A) < P(B), \\ A \sim_P B &\quad \text{if and only if} \quad P(A) = P(B). \end{aligned}$$

The ranking indices were often introduced without a clear justification and without satisfying a minimal set of conditions and, therefore, shortcomings were found for most of them, as we already pointed out in [36]. We give here only two examples, other discussions will be included later.

Example 4.1. ([36]) For a trapezoidal fuzzy number $T = (t_1, t_2, t_3, t_4)$ the parametrized ranking index M_r, $r > 0$, introduced in [89], becomes

$$M_r(T) = \left(\frac{2\left(t_1^{r+2} - t_2^{r+2}\right)(t_3 - t_4) - 2\left(t_3^{r+2} - t_4^{r+2}\right)(t_1 - t_2)}{(r+2)(r+1)(t_1 - t_2)(t_3 - t_4)(t_1 + t_2 - t_3 - t_4)} \right)^{\frac{1}{r}}.$$

It gives

$$M_2((-10,-2,-1,0)) = \left(\frac{1247}{66}\right)^{\frac{1}{2}} > \left(\frac{20}{3}\right)^{\frac{1}{2}} = M_2((1,2,3,4)),$$

and therefore $(-10,-2,-1,0) \succ_{M_2} (1,2,3,4)$, which is an obvious contradiction to our intuition. □

Example 4.2. A method of ranking fuzzy numbers with integral value was proposed in [148]. In the case of trapezoidal fuzzy numbers the ranking index reduces to

$$I_{L-W}^{\gamma}((t_1, t_2, t_3, t_4)) = \frac{1-\gamma}{2} t_1 + \frac{1-\gamma}{2} t_2 + \frac{\gamma}{2} t_3 + \frac{\gamma}{2} t_4,$$

where $\gamma \in [0,1]$ represents the degree of optimism of a decision maker. Let us consider $A = (0,1,1,2)$ and $B = \left(-\frac{4}{5}, \frac{6}{5}, \frac{6}{5}, \frac{16}{5}\right)$ and the degree of optimism $\gamma = 0.75$. Then we obtain

$$I_{L-W}^{0.75}(A) = \frac{5}{4} < \frac{17}{10} = I_{L-W}^{0.75}(B),$$

and therefore $B \succeq_{I_{L-W}^{0.75}} A$. On the other hand, because $-A = (-2,-1,-1,0)$ and $-B = \left(-\frac{16}{5}, -\frac{6}{5}, -\frac{6}{5}, \frac{4}{5}\right)$, our expectation is that $-A \succeq_{I_{L-W}^{0.75}} -B$. Nevertheless,

$$I_{L-W}^{0.75}(-A) = -\frac{3}{4} < -\frac{7}{10} = I_{L-W}^{0.75}(-B),$$

that is $-B \succeq_{I_{L-W}^{0.75}} -A$. □

Of course, the consequences of using unsuitable rankings are not only theoretical. A simple example in decision theory is given below.

Example 4.3. A decision maker is responsible for evaluating two alternatives Ω and Θ under n criteria $c_1, \ldots, c_n$. We denote by $A_i \in \mathbb{F}(\mathbb{R})$ the performance of the alternative Ω with respect to criterion c_i, by $B_i \in \mathbb{F}(\mathbb{R})$ the performance of the alternative Θ with respect to criterion c_i and by $w_i \in \mathbb{R}$ the weight of the criterion c_i,

$i \in \{1, \ldots, n\}$. We aggregate the performances of the alternatives using the weighted arithmetic mean, i.e. $A = \sum_{i=1}^{n} w_i \cdot A_i \in \mathbb{F}(\mathbb{R})$ and $B = \sum_{i=1}^{n} w_i \cdot B_i \in \mathbb{F}(\mathbb{R})$ represent the performance of Ω and Θ, respectively. Now, we introduce a new criterion c_{n+1}. The decision maker considers the performances A_{n+1} and B_{n+1} of Ω and Θ with respect to c_{n+1} to be equal, i.e. $A_{n+1} = B_{n+1}$. The performances of alternatives Ω and Θ become $\overline{A} = A + w_{n+1} \cdot A_{n+1}$ and $\overline{B} = B + w_{n+1} \cdot B_{n+1}$, where w_{n+1} is the weight of the criterion c_{n+1}. If the ranking index R is used to decide which alternative is better and $\succ_R$ does not satisfy $\mathbb{A}_5'$ and/or $\mathbb{A}_6$ then it is possible to obtain $A \succ_R B$, that is Ω is better than Θ and $\overline{B} \succ_R \overline{A}$, i.e. Θ is better than Ω, which is a contradiction with our intuition. □

Below we discuss about requirements when the order is generated by a ranking index.

Remark 4.1. In all that follows we make two natural assumptions as they are fulfilled by all (at least what we know from the existing literature) ranking approaches derived from ranking indices. Firstly, we assume that $\mathbb{R} \subset \mathscr{S}$. Then, we suppose that if $P : \mathscr{S} \to \mathbb{R}$ is a ranking index then $P\,|_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ is continuous. The first assumption can be weakened. For example, when we are interested to rank a subclass of an important family of fuzzy numbers, like the set of positive trapezoidal fuzzy numbers. In this case it suffices to assume that $\mathbb{R} \cap \mathscr{S}$ is a closed interval in $\mathbb{R}$. But this generalization will be the subject of future research. In these circumstances one can easily prove that $\mathbb{A}_4$ holds on $\mathscr{S}$ whenever $\mathbb{A}_4'$ holds on $\mathscr{S}$. In other words requirement $\mathbb{A}_4'$ is stronger than requirement $\mathbb{A}_4$.

Definition 4.1. ([36]) Two orderings $\succeq^1$ and $\succeq^2$ on the set $\mathscr{S}$ are said to be equivalent if for any $A, B \in \mathscr{S}$ $A \succeq^1 B$ results in $A \succeq^2 B$ and $A \not\succeq^1 B$ results in $A \not\succeq^2 B$.

If $\succeq^1$ and $\succeq^2$ are total binary relations (in particular those generated by ranking indices) the second requirement in the above definition is equivalent with the requirement that $A \succ^1 B$ results in $A \succ^2 B$.

Obviously, we can say that $\succeq^1$ and $\succeq^2$ generate the same ordering approach since any reasonable property $\mathbb{A}$ from the above list holds for $\succeq^1$ if and only if it holds for $\succeq^2$ too.

A particular type of ranking index is the so called defuzzifier (see, e.g., [4, 12, 171]) which is a ranking index $P : \mathscr{S} \to \mathbb{R}$ that satisfies the following requirement:

$\mathbb{A}_4''$) $P(A) \in \mathrm{supp}(A)$ for any $A \in \mathscr{S}$.

One can easily prove that the expected value (1.72) or the value (1.78) are defuzzifiers when applied on any arbitrary subset $\mathscr{S} \subseteq \mathbb{F}(\mathbb{R})$ while, in general, the ambiguity (1.79) is not a defuzzifier.

The advantage of working with defuzzifiers is obvious since they have a simple form and natural interpretation. On the other hand, we can say that it suffices to use only defuzzifiers in applications. This is so because even if $\mathbb{A}_4''$ implies $\mathbb{A}_4'$ (and implicitly $\mathbb{A}_4$), we can say that actually $\mathbb{A}_4''$ are $\mathbb{A}_4'$ are equivalent.

Theorem 4.1. *(see [36]) If $P : \mathscr{S} \to \mathbb{R}$ is a ranking index on $\mathscr{S}$ such that $\succeq_P$ satisfies $\mathbb{A}_4'$ then there exists a ranking index $R : \mathscr{S} \to \mathbb{R}$ which satisfies $\mathbb{A}_4''$ and $\succeq_R$ is equivalent to $\succeq_P$. Moreover $R = P_{\mathbb{R}}^{-1} \circ P$, where $P_{\mathbb{R}}^{-1} : P(\mathbb{R}) \to \mathbb{R}$, is the inverse of the function $P_{\mathbb{R}} : \mathbb{R} \to P(\mathbb{R})$ and $P_{\mathbb{R}}(x) = P\mid_{\mathbb{R}}(x) = P(x)$ (basically $P_{\mathbb{R}}$ and $P\mid_{\mathbb{R}}$ represent the same function if we disregard their domains). In addition, $P_{\mathbb{R}}^{-1}$ is strictly increasing and continuous.*

Proof. Let us choose arbitrarily $A \in \mathscr{S}$ and suppose that $\text{supp}(A) = [a,b]$. Since $\succeq_P$ satisfies $\mathbb{A}_4'$ (and implicitly $\mathbb{A}_4$) it results in $P(a) \leq P(A) \leq P(b)$. The continuity of $P\mid_{\mathbb{R}}$ (see Remark 4.1) implies that there exists $x_A \in [a,b]$ such that $P(x_A) = P(A)$. In addition, x_A is unique with this property. Indeed, for any $x \in [a,b]$, $x \neq x_A$, either $P(x_A) < P(x)$ or $P(x_A) > P(x)$ because otherwise requirement $\mathbb{A}_4'$ is violated. Therefore, we can define the ranking index $R : \mathscr{S} \to \mathbb{R}, R(A) = x_A$ which satisfies requirement $\mathbb{A}_4''$. The equivalence between $\succeq_P$ and $\succeq_R$ is immediate by the construction of R.

For the rest of the proof let us note at first that since $\succeq_P$ satisfies $\mathbb{A}_4'$ it is immediate that $P_{\mathbb{R}}$ is strictly increasing and hence bijective. Therefore, since $P_{\mathbb{R}}$ is also continuous then $P_{\mathbb{R}}^{-1}$ is strictly increasing and continuous. Let us note that the function $P_{\mathbb{R}}^{-1} \circ P$ is correctly defined since by the first part of this proof we also get $P(\mathbb{R}) = P(\mathbb{F}(\mathbb{R}))$. Finally, for any $A \in \mathscr{S}$ we have

$$R(A) = x_A = P_{\mathbb{R}}^{-1}(P(x_A)) = P_{\mathbb{R}}^{-1}(P(A)).$$

and the proof is complete. ■

Remark 4.2. Requirements $\mathbb{A}_4$ and $\mathbb{A}_4'$ are natural for any ordering between fuzzy numbers. Their absence lead to shortcomings and any subsequent discussion could be stopped. Therefore (taking into account Remark 4.1 and Theorem 4.1) we can focus on ranking indices which satisfy $\mathbb{A}_4''$.

Below we apply Theorem 4.1 for two known ranking indices.

Example 4.4. (see [36]) A ranking index based on centroids of fuzzy numbers was introduced in [69]. It reduces to

$$P_{Ch-T}(A) = \frac{(a+b+c)(a+4b+c)}{9(a+2b+c)},$$

for every $A = (a,b,c) \in \mathbb{F}^{\Delta}(\mathbb{R})$ and satisfies $\mathbb{A}_4'$ on $\mathscr{S} = \mathbb{F}^{\Delta}(\mathbb{R})$. Moreover, $P_{Ch-T}\mid_{\mathbb{R}}$: $\mathbb{R} \to \mathbb{R}$ is continuous. From $P_{Ch-T}(x_A) = P_{Ch-T}(A)$ we obtain $x_A = 2P_{Ch-T}(A)$. according to Theorem 4.1 the ranking $\succeq_{P_{Ch-T}}$ is equivalent to $\succeq_{R_{Ch-T}}$, where R_{Ch-T} : $\mathbb{F}^{\Delta}(\mathbb{R}) \to \mathbb{R}$ is given by

$$R_{Ch-T}(A) = 2P_{Ch-T}(A)$$

and satisfies $\mathbb{A}_4''$. □

Example 4.5. (see [36]) The ranking index introduced in [67] was intensely cited ([4, 12, 88, 90, 91, 184], etc.). It was introduced in a more general framework, but it becomes

$$P_{Cho-Li}(A) = \frac{1}{2(M-m)}\left(\int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha - 2m\right),$$

for every $A \in \mathbb{F}(\mathbb{R})$, where $m, M \in \mathbb{R}$, $M \neq m$, reflect decision maker's inclination or aversion to risk. It is immediate that $\succeq_{P_{Cho-Li}}$ satisfies $\mathbb{A}_4'$ and $P_{Cho-Li}|_{\mathbb{R}}$ is continuous. By Theorem 4.1 and its proof we get

$$R_{Cho-Li}(A) = \frac{1}{2}\int_0^1 A_L(\alpha)d\alpha + \frac{1}{2}\int_0^1 A_U(\alpha)d\alpha,$$

which satisfies $\mathbb{A}_4''$. Moreover, $\succeq_{R_{Cho-Li}}$ is equivalent to $\succeq_{P_{Cho-Li}}$. But since $R_{Cho-Li} = EV$ therefore, at least for fuzzy numbers in the sense of the present paper, we obtain the same ranking as the one generated by the expected value (1.72), often used in applications (see, e.g., [23, 68]). □

It is worth noting that Table 1 in [184] offers a list of other ranking indices for which Theorem 4.1 is applicable.

4.3 Characterization of effective ranking indices

In this section we characterize a family of valuable ranking indices on a set $\mathscr{S} \subseteq \mathbb{F}(\mathbb{R})$ satisfying suitable properties with respect to requirements $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_4$, $\mathbb{A}_4'$, $\mathbb{A}_5$, $\mathbb{A}_5'$, $\mathbb{A}_6$ and $\mathbb{A}_4''$.

Theorem 4.2. *([36]) Consider $\mathscr{S} \subseteq \mathbb{F}(\mathbb{R})$ such that $\mathscr{S} + \mathscr{S} \subseteq \mathscr{S}$ and a ranking index $R : \mathscr{S} \to \mathbb{R}$ which satisfies $\mathbb{A}_4''$. If $\succeq_R$ satisfies $\mathbb{A}_5$ on $\mathscr{S}$ then R is additive on $\mathscr{S}$. Moreover, $\succeq_R$ satisfies $\mathbb{A}_5'$ on $\mathscr{S}$.*

Proof. The requirements $\mathbb{A}_2$ and $\mathbb{A}_3$ are satisfied on $\mathscr{S}$ because $\succeq_R$ is generated by a ranking index. Since R satisfies $\mathbb{A}_4''$ on $\mathscr{S}$ it results in $R(R(A)) = R(A)$, for all $A \in \mathscr{S}$, which implies that $A \sim_R R(A)$ for all $A \in \mathscr{S}$. Let us now choose arbitrarily $A, B \in \mathscr{S}$. We have $A \sim_R R(A)$ which implies, by $\mathbb{A}_2$ and $\mathbb{A}_5$, that $A - R(A) \sim_R 0$ and then again, by $\mathbb{A}_2$ and $\mathbb{A}_5$, we get that $A + B - R(A) - R(B) \sim_R B - R(B)$. Since $B - R(B) \sim_R 0$, by $\mathbb{A}_2$ and $\mathbb{A}_3$, we get $A + B - R(A) - R(B) \sim_R 0$ and applying again $\mathbb{A}_2$ and $\mathbb{A}_5$ we obtain $A + B \sim_R R(A) + R(B)$. This implies $R(A+B) = R(R(A) + R(B))$. By $\mathbb{A}_4''$ we obtain $R(R(A) + R(B)) = R(A) + R(B)$ which results in $R(A+B) = R(A) + R(B)$. Now, since R is additive on $\mathscr{S}$ it is immediate that $\mathbb{A}_5'$ is satisfied by $\succeq_R$ on $\mathscr{S}$. ■

The above result help us to obtain some negative examples with respect to $\mathbb{A}_5$ from the non-additivity of the restrictions of the ranking indices to suitable families of fuzzy numbers. We exemplify it by the following ranking index.

Example 4.6. (see [36]) The restriction of the ranking index introduced in [144] (see also [179]) to trapezoidal fuzzy numbers becomes $R_{Lee-Li} : \mathbb{F}^T(\mathbb{R}) \to \mathbb{R}$ given as follows

$$R_{Lee-Li}(A) = \frac{-a^2 - b^2 + c^2 + d^2 - ab + cd}{3(-a-b+c+d)},$$

for every $A = (a,b,c,d) \in \mathbb{F}^T(\mathbb{R})$. One can verify that the property $\mathbb{A}_4''$ holds. Because

$$\begin{aligned} &R_{Lee-Li}((1,2,3,4)) + R_{Lee-Li}((2,3,4,6)) = \frac{5}{2} + \frac{19}{5} \\ &\neq \frac{170}{27} = R_{Lee-Li}((3,5,7,10)) = R_{Lee-Li}((1,2,3,4) + (2,3,4,6)), \end{aligned}$$

by Theorem 4.2 we obtain that $\succeq_{R_{Lee-Li}}$ does not satisfy $\mathbb{A}_5$ on $\mathbb{F}^T(\mathbb{R})$ and implicitly on $\mathbb{F}(\mathbb{R})$. □

Analyzing carefully the proof of Theorem 4.2 we obtain an equivalence between $\mathbb{A}_5$ and $\mathbb{A}_5'$ as follows.

Corollary 4.1. *([36]) Consider $\mathscr{S} \subseteq \mathbb{F}(\mathbb{R})$ such that $\mathscr{S} + \mathscr{S} \subseteq \mathscr{S}$ and a ranking index $R : \mathscr{S} \to \mathbb{R}$ which satisfies $\mathbb{A}_4''$. The order $\succeq_R$ on $\mathscr{S}$ satisfies $\mathbb{A}_5$ on $\mathscr{S}$ if and only if it satisfies $\mathbb{A}_5'$ on $\mathscr{S}$.*

Proof. Let us observe that the reasoning in the proof of Theorem 4.2 is not influenced at all if instead of $\mathbb{A}_5$ we use everywhere $\mathbb{A}_5'$. For example, looking on the proof of Theorem 4.2 we conclude that $\mathbb{A}_2$ and $\mathbb{A}_5$ lead to $A - R(A) \sim_R 0$. But this also holds if instead of $\mathbb{A}_5$ we use $\mathbb{A}_5'$. Suppose, contrary to our claim, that $A - R(A) \sim_R 0$ does not hold. Then either $A - R(A) \prec_R 0$ or $A - R(A) \succ_R 0$. In the first case, by $\mathbb{A}_5'$ we obtain $A \prec_R R(A)$ and this obviously is a contradiction. Similarly, we obtain a contradiction in the second case too. Thus, we must have $A - R(A) \sim_R 0$. Similarly, everywhere in the proof of Theorem 4.2 we can replace the assumption that $\succeq_R$ satisfies $\mathbb{A}_5$ on $\mathscr{S}$ by the assumption that $\succeq_R$ satisfies $\mathbb{A}_5'$ on $\mathscr{S}$. ■

Corollary 4.2. *([36]) Consider $\mathscr{S} \subseteq F(\mathbb{R})$ such that $\mathscr{S} + \mathscr{S} \subseteq \mathscr{S}$. If $R : \mathscr{S} \to \mathbb{R}$ is a ranking index such that $\mathbb{A}_4'$ and $\mathbb{A}_5$ are satisfied by $\succeq_R$ on $\mathscr{S}$, then $\mathbb{A}_5'$ is also satisfied by $\succeq_R$ on $\mathscr{S}$. Moreover, there exists an additive ranking index $R_* : \mathscr{S} \to \mathbb{R}$ which satisfies $\mathbb{A}_4''$ on $\mathscr{S}$ and $\succeq_{R_*}$ is equivalent to $\succeq_R$.*

Proof. Since R satisfies $\mathbb{A}_4'$, by Theorem 4.1, there exists $R_* : \mathscr{S} \to \mathbb{R}$ which satisfies $\mathbb{A}_4''$ and such that the order $\succeq_{R_*}$ is equivalent to $\succeq_R$. Consequently, $\mathbb{A}_5$ is satisfied by $\succeq_{R_*}$ on $\mathscr{S}$. Substituting R by R_* in Theorem 4.2 we conclude that R_* is additive and that $\succeq_{R_*}$ satisfies $\mathbb{A}_5'$ on $\mathscr{S}$. Since $\succeq_R$ is equivalent to $\succeq_{R_*}$ it yields that $\succeq_R$ satisfies $\mathbb{A}_5'$ on $\mathscr{S}$ and hence the proof is complete. ■

Example 4.7. By Corollary 4.2, if we take $R = P_{Cho-Li}$ considered in Example 4.5 then $R_* = EV$. □

Passing to scalar multiplication we obtain a result corresponding to that given in Theorem 4.2.

Theorem 4.3. *([36]) Consider $\mathscr{S} \subseteq \mathbb{F}(\mathbb{R})$ such that $\lambda \cdot \mathscr{S} \subseteq \mathscr{S}$ for all $\lambda \in \mathbb{R}$ and suppose that $R : \mathscr{S} \to \mathbb{R}$ is a ranking index which satisfies $\mathbb{A}_4''$. If $\succeq_R$ satisfies $\mathbb{A}_6$ on $\mathscr{S}$ then R is scale invariant, i.e. $R(\lambda \cdot A) = \lambda R(A)$, for any $\lambda \in \mathbb{R}$ and $A \in \mathscr{S}$.*

Proof. Again, we begin the proof by noticing that $\mathbb{A}_1, \mathbb{A}_2$ and $\mathbb{A}_3$ are satisfied by $\succeq_R$ on $\mathscr{S}$. Now, let us choose arbitrarily $A \in \mathscr{S}$ and $\lambda \in \mathbb{R}$. We have $A \sim_R R(A)$ which by $\mathbb{A}_2$ and $\mathbb{A}_6$ implies that $\lambda \cdot A \sim_R \lambda R(A)$. Therefore, we obtain $R(\lambda \cdot A) = R(\lambda R(A))$ and since obviously $R(\lambda R(A)) = \lambda R(A)$ we get $R(\lambda \cdot A) = \lambda R(A)$. ■

Theorem 4.3 indicates a method to prove that $\mathbb{A}_6$ is not satisfied. Actually, by choosing a suitable family of fuzzy numbers on which the ranking index is not scale invariant we immediately obtain that the generated order does not satisfy $\mathbb{A}_6$. Let us consider the following example.

Example 4.8. We restrict the ranking index introduced in [148] to triangular fuzzy numbers. It becomes $R_{L-W}^{\theta} : \mathbb{F}^{\Delta}(\mathbb{R}) \to \mathbb{R}$ given by

$$R_{L-W}^{\gamma}(A) = \frac{1-\gamma}{2}a + \frac{1}{2}b + \frac{\gamma}{2}c,$$

for every $A = (a,b,c) \in \mathbb{F}^{\Delta}(\mathbb{R})$, where $\gamma \in [0,1]$ reflects the degree of optimism of a decision maker. Since

$$\begin{aligned} -R_{L-W}^{\theta}(0,1,2) &= -\theta - \frac{1}{2} \\ &\neq \theta - \frac{3}{2} = R_{L-W}^{\theta}(-2,-1,0) = R_{L-W}^{\theta}(-(0,1,2)) \end{aligned}$$

for every $\theta \neq \frac{1}{2}$, by Theorem 4.3, we obtain that $\succeq_{R_{L-W}^{\theta}}$ does not satisfy $\mathbb{A}_6$ on $\mathbb{F}^{\Delta}(\mathbb{R})$ and implicitly on $\mathbb{F}(\mathbb{R})$. □

Combining Theorems 4.1 and 4.3 we obtain the following conclusion.

Corollary 4.3. *([36]) Consider $\mathscr{S} \subseteq \mathbb{F}(\mathbb{R})$ such that $\lambda \cdot \mathscr{S} \subseteq \mathscr{S}$ for all $\lambda \in \mathbb{R}$ and suppose that $R : \mathscr{S} \to \mathbb{R}$ is a ranking index. If $\succeq_R$ satisfies $\mathbb{A}_4'$ and $\mathbb{A}_6$ on $\mathscr{S}$, then there exists a scale invariant ranking index $R_* : \mathscr{S} \to \mathbb{R}$ which satisfies $\mathbb{A}_4''$ on $\mathscr{S}$ and which generates on $\mathscr{S}$ an equivalent order with $\succeq_R$.*

Any reasonable ranking between fuzzy numbers from a set $\mathscr{S}$ should satisfy the requirements discussed in Section 4.2. Therefore, let us adopt the following notation

$$M(\mathscr{S}) = \left\{P : \mathscr{S} \to \mathbb{R} \ | \ \succeq_P \text{ satisfies } \mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \mathbb{A}_4, \mathbb{A}_4', \mathbb{A}_5, \mathbb{A}_5', \mathbb{A}_6\right\}.$$

Taking into account Theorem 4.1 we also consider (see also ([36]))

$$M_*(\mathscr{S}) = \left\{P : \mathscr{S} \to \mathbb{R} \ | \ P \text{ satisfies } \mathbb{A}_4'' \right. \\ \left. \text{and } \succeq_P \text{ satisfies } \mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \mathbb{A}_5, \mathbb{A}_5', \mathbb{A}_6\right\}.$$

It is clear that in general $M_*(\mathscr{S}) \subset M(\mathscr{S})$. From Theorem 4.1 we also know that if $P \in M(\mathscr{S})$ then there exists $P_* \in M_*(\mathscr{S})$ such that $\succeq_{P_*}$ is equivalent to $\succeq_P$ on $\mathscr{S}$. Therefore, to find effective orders over $\mathscr{S}$ it suffices to study the elements of $M_*(\mathscr{S})$. This observation simplifies the whole procedure since $\mathbb{A}_4''$ may reduce the calculations part - see Sections 4.4 and 4.5 for the cases $\mathscr{S} = \mathbb{F}^{\Delta}(\mathbb{R})$ and $\mathscr{S} = \mathbb{F}^T(\mathbb{R})$, respectively. We conclude this section with some characterizations of the elements belonging to the classes $M(\mathscr{S})$ and $M_*(\mathscr{S})$.

Theorem 4.4. *([36]) Consider $\mathscr{S} \subseteq \mathbb{F}(\mathbb{R})$ such that $\mathscr{S} + \mathscr{S} \subseteq \mathscr{S}$ and $\lambda \cdot \mathscr{S} \subseteq \mathscr{S}$ for all $\lambda \in \mathbb{R}$ and a ranking index $P : \mathscr{S} \to \mathbb{R}$. Then we have:*

(i) $P \in M_*(\mathscr{S})$ if and only if P satisfies $\mathbb{A}_4''$ on $\mathscr{S}$ and P is linear on $\mathscr{S}$;
(ii) $P \in M(\mathscr{S})$ if and only if there exists $P_* \in M_*(\mathscr{S})$ such that $\succeq_P$ and $\succeq_{P_*}$ are equivalent on $\mathscr{S}$.

Proof. (i) We prove only the direct implication since the inverse implications are immediate. Since $P \in M_*(\mathscr{S})$ then $\mathbb{A}_4''$ obviously holds. This, together with the simple observation that $\mathbb{A}_5$ and $\mathbb{A}_6$ hold, lead to the conclusion that the assumptions of Theorems 4.2 and 4.3 are fulfilled and hence we get the linearity of P.

(ii) The existence of P_* is guaranteed by Theorem 4.1 . Since $\succeq_P$ and $\succeq_{P_*}$ are equivalent on $\mathscr{S}$ and since $P \in M(\mathscr{S})$, we get $P_* \in M_*(\mathscr{S})$. To prove the inverse implication it suffices to notice that obviously by $P_* \in M_*(\mathscr{S})$ we get $P_* \in M(\mathscr{S})$ and the desired conclusion follows easily by taking into account that $\succeq_P$ and $\succeq_{P_*}$ are equivalent on $\mathscr{S}$. ■

4.4 Characterization of valuable ranking indices on triangular fuzzy numbers

Since triangular fuzzy numbers and trapezoidal fuzzy numbers have a simple representation and they are so often used in applications, when discussing on ranking fuzzy numbers we should investigate as much as possible these families of fuzzy numbers. In addition, almost all numerical examples in the literature dedicated to the study of ranking of fuzzy numbers are performed just on triangular or trapezoidal fuzzy numbers. There are also such papers that discuss only the ranking of triangular or trapezoidal fuzzy numbers (see, e.g., [4, 90]). That is why below we study the ranking indices on $\mathbb{F}^{\Delta}(\mathbb{R})$ and the next section is dedicated to ranking indices on $\mathbb{F}^{T}(\mathbb{R})$.

We consider the α-cut of a triangular fuzzy number Δ in the form (see [4])

$$\Delta_\alpha = [x_0 - \sigma + \sigma\alpha, x_0 + \beta - \beta\alpha],$$

where $x_0, \sigma, \beta \in \mathbb{R}, \sigma \geq 0, \beta \geq 0$. Let us denote by $\Delta = [x_0, \sigma, \beta]$ a such fuzzy number. By (1.12) we have

$$t_1 = x_0 - \sigma \tag{4.1}$$
$$t_2 = x_0 \tag{4.2}$$
$$t_3 = x_0 + \beta. \tag{4.3}$$

After some simple calculations, by (1.72), (1.79) and (1.78) we get

$$EV(\Delta) = x_0 - \frac{1}{4}\sigma + \frac{1}{4}\beta, \tag{4.4}$$
$$Amb(\Delta) = \frac{1}{6}\sigma + \frac{1}{6}\beta, \tag{4.5}$$
$$Val(\Delta) = x_0 - \frac{1}{6}\sigma + \frac{1}{6}\beta. \tag{4.6}$$

The expected value, value and linear combinations of ambiguity and value are considered as ranking indices (see [80, 191]), therefore in [35] the following set of ranking indices is explored

$$\Omega^{\Delta} = \{R : \mathbb{F}^{\Delta}(\mathbb{R}) \to \mathbb{R} \mid R([x_0, \sigma, \beta]) = ax_0 + b\sigma + c\beta\}.$$

It is proved that there are no ranking indices in $M_*(\mathbb{F}^{\Delta}(\mathbb{R}))$ and $M(\mathbb{F}^{\Delta}(\mathbb{R}))$ which do not belong to $\Omega^{\triangle}$. Since the order $\succeq_R$, where $R \in \Omega^{\Delta}$, is generated by a ranking index then conditions $\mathbb{A}_1$, $\mathbb{A}_2$ and $\mathbb{A}_3$ hold. One can easily prove that $R(\Delta + \Delta') = R(\Delta) + R(\Delta')$ for all $R \in \Omega^{\Delta}$ and $\Delta, \Delta' \in \mathbb{F}^{\Delta}(\mathbb{R})$. Hence $\mathbb{A}_5$ and $\mathbb{A}_5'$ hold too. Therefore, the necessary and sufficient conditions for the constants a, b and c such that $R \in \Omega^{\Delta}$ satisfies $\mathbb{A}_4$, $\mathbb{A}_4'$, $\mathbb{A}_4''$ or $\mathbb{A}_6$ are determined.

Theorem 4.5. *([35]) Let $R \in \Omega^{\triangle}$, where $R([x_0, \sigma, \beta]) = ax_0 + b\sigma + c\beta$. The order $\succeq_R$ satisfies $\mathbb{A}_4$ on $\mathbb{F}^{\Delta}(\mathbb{R})$ if and only if*

$$a \geq c \geq 0 \tag{4.7}$$

and

$$a \geq -b \geq 0. \tag{4.8}$$

Proof. ($\Rightarrow$) We consider particular cases of $\Delta = [x_0, \sigma, \beta]$ and $\Delta' = [x_0', \sigma', \beta']$ such that $\inf \operatorname{supp}(\Delta) \geq \sup \operatorname{supp}(\Delta')$ is satisfied, until we obtain that (4.7) and (4.8) hold. Note that assuming $\mathbb{A}_4$ we have $\Delta \succeq_R \Delta' \Leftrightarrow R(\Delta) \geq R(\Delta')$.

Firstly, let us consider a particular case when $x_0 > x'_0 > 0$ and $\sigma = \beta = \sigma' = \beta' = 0$. Since $R(\Delta) \geq R(\Delta')$ implies $ax_0 \geq ax'_0$ and since $x_0 > x'_0 > 0$, it is immediate that $a \geq 0$.

Suppose $x_0 = \sigma = \beta = x'_0 = \beta' = 0$. Then $R(\Delta) \geq R(\Delta')$ implies $b\sigma' \leq 0$ and hence we obtain $b \leq 0$. Consider now the case when $x_0 = \sigma = x'_0 = \sigma' = \beta' = 0$ and $\beta > 0$. Since $R(\Delta) \geq R(\Delta')$ then $c\beta \geq 0$, so we get $c \geq 0$.

Now, let us assume that $x_0 = \beta' = 1$ and $\sigma = x'_0 = \sigma' = \beta = 0$. By $R(\Delta) \geq R(\Delta')$ it is immediate that we get $a - c \geq 0$. Finally, we consider the case when $x_0 = \sigma = 1$ and $x'_0 = \sigma' = \beta' = \beta = 0$. By $R(\Delta) \geq R(\Delta')$ we get $a + b \geq 0$.

Collecting the inequalities obtained in the particular cases considered above we obtain that (4.7) and (4.8) hold.

($\Leftarrow$) Let a, b, c be real numbers satisfying (4.7) and (4.8). Moreover, let $\Delta = [x_0, \sigma, \beta]$ and $\Delta' = [x'_0, \sigma', \beta']$ denote two arbitrary triangular fuzzy numbers such that $\inf \operatorname{supp}(\Delta) \geq \sup \operatorname{supp}(\Delta')$. This immediately implies $x_0 - x'_0 \geq \sigma + \beta' \geq 0$. By our assumption $-b\sigma' \geq 0$ and $c\beta \geq 0$, hence by direct calculations we get

$$\begin{aligned} R(\Delta) - R(\Delta') &= a(x_0 - x'_0) + b(\sigma - \sigma') + c(\beta - \beta') \\ &\geq a(\sigma + \beta') + b\sigma - c\beta' \\ &= \sigma(a + b) + \beta'(a - c) \geq 0. \end{aligned}$$

This implies $\Delta \succeq_R \Delta'$ and the theorem is proved. ■

Theorem 4.6. *([35]) Let $R \in \Omega^{\triangle}$ such that $R([x_0, \sigma, \beta]) = ax_0 + b\sigma + c\beta$. The order $\succeq_R$ satisfies $\mathbb{A}'_4$ if and only if*

$$a \geq c \geq 0 \tag{4.9}$$

$$a \geq -b \geq 0 \tag{4.10}$$

and

$$a > 0. \tag{4.11}$$

Proof. ($\Rightarrow$) By Remark 4.1 we know that if $\mathbb{A}'_4$ is satisfied by $\succeq_R$ then $\mathbb{A}_4$ is satisfied by $\succeq_R$. Since $\mathbb{A}_4$ holds, by the previous theorem it suffices to prove that $a > 0$. For this purpose let us consider a particular case of $\Delta = [x_0, \sigma, \beta]$ and $\Delta' = [x'_0, \sigma', \beta']$ when $x_0 = 1$ and $\sigma = \beta = x'_0 = \sigma' = \beta' = 0$. Clearly, we have $\inf \operatorname{supp}(\Delta) > \sup \operatorname{supp}(\Delta')$, and therefore $R(\Delta) > R(\Delta')$, i.e. $a > 0$.

($\Leftarrow$) Let a, b, c be real numbers such that (4.9)-(4.11) are satisfied. Let $\Delta = [x_0, \sigma, \beta]$ and $\Delta' = [x'_0, \sigma', \beta']$ denote two arbitrary triangular fuzzy numbers such that $\inf \operatorname{supp}(\Delta) > \sup \operatorname{supp}(\Delta')$. It is easy to check that $x_0 - x'_0 > \sigma + \beta' \geq 0$. Then, since $a > 0$, we obtain

$$a(x_0 - x'_0) > a(\sigma + \beta')$$

which implies that

$$\begin{aligned} R(\Delta) - R(\Delta') &= a(x_0 - x_0') + b(\sigma - \sigma') + c(\beta - \beta') \\ &> a(\sigma + \beta') + b(\sigma - \sigma') + c(\beta - \beta') \\ &\geq a(\sigma + \beta') + b\sigma - c\beta' \\ &= \sigma(a+b) + \beta'(a-c) \geq 0. \end{aligned}$$

We obtain $R(\Delta) > R(\Delta')$ and therefore $\Delta \succ_R \Delta'$, which completes the proof. ■

Theorem 4.7. *([35]) The ranking index $R \in \Omega^{\triangle}$ of the form $R([x_0, \sigma, \beta]) = ax_0 + b\sigma + c\beta$ satisfies $\mathbb{A}_4''$ if and only if*

$$a = 1 \tag{4.12}$$

$$b \in [-1, 0] \tag{4.13}$$

$$c \in [0, 1]. \tag{4.14}$$

Proof. ($\Rightarrow$) Let us notice that if $\mathbb{A}_4''$ holds then it is immediate that $\mathbb{A}_4'$ holds too. Therefore, comparing conditions (4.9)-(4.11) and (4.12)-(4.14), it follows that for the direct implication of the present theorem it suffices to prove $a = 1$. Since we have supposed that $\mathbb{A}_4''$ holds, it results that for any triangular fuzzy number $\Delta = [x_0, \sigma, \beta]$ we have

$$x_0 - \sigma \leq ax_0 + b\sigma + c\beta \leq x_0 + \beta. \tag{4.15}$$

Substitution $x_0 = 1$ and $\sigma = \beta = 0$ in (4.15) we get $1 \leq a \leq 1$ and thus we obtain (4.12).

($\Leftarrow$) Let a, b, c be real numbers such that (4.12)-(4.14) are satisfied and let $\Delta = [x_0, \sigma, \beta]$ denote a triangular fuzzy number. Conditions (4.12)-(4.14) imply

$$R(\Delta) = x_0 + b\sigma + c\beta \leq x_0 + c\beta \leq x_0 + \beta$$

and

$$R(\Delta) = x_0 + b\sigma + c\beta \geq x_0 + b\sigma \geq x_0 - \sigma.$$

From these two inequalities we get $R(\Delta) \in \operatorname{supp}\Delta$ and the theorem is proved. ■

Theorem 4.8. *([35]) Let $R \in \Omega^{\triangle}$ be such that $R([x_0, \sigma, \beta]) = ax_0 + b\sigma + c\beta$. The order $\succeq_R$ satisfies $\mathbb{A}_6$ if and only if*

$$b + c = 0. \tag{4.16}$$

Proof. ($\Rightarrow$) Let us consider $\Delta = [0, 1, 1]$ and $O = [0, 0, 0]$. If $R(\Delta) \geq R(O)$ then $R(-\Delta) \leq R(O)$. Since $R(O) = 0$ and $-\Delta = \Delta$ it follows immediately that $R(\Delta) = 0$, so $b + c = 0$. If $R(\Delta) \leq R(O)$ then the reasoning is similar and we omit the details.

($\Leftarrow$) Let us consider the reals a, b, c such that $b + c = 0$. If $\Delta = [x_0, \sigma, \beta]$ then $R(\Delta) = ax_0 - c\sigma + c\beta$. To prove that $\mathbb{A}_6$ holds it suffices to show that the operator R is scale invariant. If $\lambda \geq 0$ then one can easily prove that $R(\lambda \cdot \Delta) = \lambda R(\Delta)$. Since

$$R(-\Delta) = -ax_0 - c\beta + c\sigma = -(ax_0 - c\sigma + c\beta) = -R(\Delta),$$

we immediately obtain $R(\lambda \cdot \Delta) = \lambda R(\Delta)$ for any $\Delta \in \mathbb{F}^{\Delta}(\mathbb{R})$ and $\lambda \in \mathbb{R}$. ■

Example 4.9. Delgado, Vila and Voxman [80] proposed a ranking index given by

$$ri(\lambda, \delta)(A) = \lambda Val(A) + \delta Amb(A),$$

where $\lambda \in [0,1]$ and $\delta \in [-1,1]$ are such that $|\delta| \ll \lambda$. Coefficient δ represents the decision-maker's attitude to uncertainty. If $\Delta = [x_0, \sigma, \beta]$ is a triangular fuzzy number then

$$ri(\lambda, \delta)(\Delta) = \lambda x_0 + \frac{\delta - \lambda}{6}\sigma + \frac{\lambda + \delta}{6}\beta.$$

It is easily seen that $ri(\lambda, \delta) \in \Omega^{\triangle}$ for all λ and δ. By Theorems 4.5 and 4.6 we conclude that $\succeq_{ri(\lambda,\delta)}$ satisfies $\mathbb{A}_4$ and $\mathbb{A}_4'$ for every λ and δ. On the other hand by Theorems 4.7 and 4.8 we conclude that $\succeq_{ri(\lambda,\delta)}$ satisfies $\mathbb{A}_4''$ if and only if $\lambda = 1$ and $\mathbb{A}_6$ if and only if $\delta = 0$. □

The following two theorems are immediate consequences of the above results. Characterizations of the elements of $\Omega^{\triangle}$ which are in $M_*(\mathbb{F}^{\triangle}(\mathbb{R}))$ and $M(\mathbb{F}^{\triangle}(\mathbb{R}))$ are presented.

Corollary 4.4. *([35]) Let $R \in \Omega^{\triangle}$ be such that $R([x_0, \sigma, \beta]) = ax_0 + b\sigma + c\beta$. Then $R \in M_*(\mathbb{F}^{\triangle}(\mathbb{R}))$ if and only if*

$$a = 1 \tag{4.17}$$

$$b + c = 0 \tag{4.18}$$

and

$$c \in [0,1]. \tag{4.19}$$

Corollary 4.5. *([35]) Let $R \in \Omega^{\triangle}$ be such that $R([x_0, \sigma, \beta]) = ax_0 + b\sigma + c\beta$. Then $R \in M(\mathbb{F}^{\triangle}(\mathbb{R}))$ if and only if $a > 0$, $b = -c$ and $a \geq c \geq 0$.*

Example 4.10. With respect to the ranking index considered in Example 4.9 we deduce that $ri(\lambda, \delta) \in M(\mathbb{F}^{\triangle}(\mathbb{R}))$ and hence $\succeq_{ri(\lambda,\delta)}$ satisfies $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_4$, $\mathbb{A}_4'$, $\mathbb{A}_5$, $\mathbb{A}_5'$ and $\mathbb{A}_6$ if and only if $\delta = 0$. Moreover, if $\lambda < 1$ then $ri(\lambda, \delta) \in M(\mathbb{F}^{\triangle}(\mathbb{R})) \setminus M_*(\mathbb{F}^{\triangle}(\mathbb{R}))$ and if $\lambda = 1$ then $ri(1,0) = Val \in M_*(\mathbb{F}^{\triangle}(\mathbb{R}))$. □

According to Corollary 4.4 it is easy to deduce that some already introduced ranking indices are elements of $M_*(\mathbb{F}^{\triangle}(\mathbb{R}))$, i.e. they satisfy $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_4$, $\mathbb{A}_4'$, $\mathbb{A}_5$, $\mathbb{A}_5'$ and $\mathbb{A}_6$. The following examples are given in [35].

Example 4.11. Let us consider a function $EV : F^{\Delta}(\mathbb{R}) \to \mathbb{R}$ which for any triangular fuzzy number $\Delta = [x_0, \sigma, \beta]$ associates its expected value (1.72), i.e. $EV(T) = x_0 - \frac{1}{4}\sigma + \frac{1}{4}\beta$. It is immediate that $EV \in M_*(\mathbb{F}^{\triangle}(\mathbb{R}))$. □

Example 4.12. In [84] a ranking procedure is proposed via the so called valuation functions. The authors consider a strictly monotonous function (valuation) $f : [0,1] \to [0,\infty)$ and a ranking index $R : \mathbb{F}^{\Delta}(\mathbb{R}) \to \mathbb{R}$ which on triangular fuzzy numbers becomes

$$R([x_0, \sigma, \beta]) = (2-\omega)x_0 - \frac{1-\omega}{2}\sigma + \frac{1-\omega}{2}\beta,$$

where

$$\omega = \frac{\int_0^1 \alpha f(\alpha)d\alpha}{\int_0^1 f(\alpha)d\alpha}.$$

Since $0 < \omega < 1$ for any valuation f, by Theorems 4.5, 4.6 and 4.8, we easily see that R generates an order which satisfies $\mathbb{A}_4$, $\mathbb{A}_4'$ and $\mathbb{A}_6$ and hence $R \in M(\mathbb{F}^{\triangle}(\mathbb{R}))$ for any valuation f. On the other hand, by Theorem 4.7 we observe that $R \notin M_*(\mathbb{F}^{\triangle}(\mathbb{R}))$. It is easily seen that $R_* \in M_*(\mathbb{F}^{\triangle}(\mathbb{R}))$, where $R_* = \frac{1}{2-\omega}R$ and $\succeq_{R_*}$ is equivalent to $\succeq_R$. Moreover, for $f(\alpha) = \alpha$, $\alpha \in [0,1]$ we obtain the ranking index considered in Example 4.9, i.e. $R_* = ri(1,0) = Val$. □

It would be important to know whether there exists any other ranking index $R \in M_*(F^{\triangle}(\mathbb{R}))$ which does not belong to $\Omega^{\triangle}$. The answer to this question is negative as the following theorem proves.

Theorem 4.9. *([35]) Let us consider a ranking index $R : \mathbb{F}^{\Delta}(\mathbb{R}) \to \mathbb{R}$. Then $R \in M_*(\mathbb{F}^{\triangle}(\mathbb{R}))$ if and only if there exists $c \in [0,1]$ such that for some $\Delta \in \mathbb{F}^{\Delta}(\mathbb{R})$, where $\Delta = [x_0, \sigma, \beta]$, we have*

$$R(\Delta) = x_0 - c\sigma + c\beta. \tag{4.20}$$

Proof. Taking into account Corollary 4.4 it is easily seen that we can obtain the desired conclusion by proving that $R \in M_*(\mathbb{F}^{\triangle}(\mathbb{R}))$ implies $R \in \Omega^{\triangle}$. Firstly, let us observe that by Theorem 22 in [36] R is linear on $\mathbb{F}^{\Delta}(\mathbb{R})$. Let $\Delta \in \mathbb{F}^{\Delta}(\mathbb{R})$ such that $\Delta = (t_1, t_2, t_3)$. Here we use (1.12) representation of fuzzy numbers because it is more suitable for this proof. Now, let us consider the triangular fuzzy numbers

$$v_1 = (0,0,1),$$
$$v_2 = (0,1,1),$$
$$v_3 = (1,1,1).$$

Having in mind the addition and the scalar multiplication of fuzzy numbers we get that $\Delta = t_1v_3 + (t_2 - t_1)v_2 + (t_3 - t_2)v_1$. The linearity of R implies $R(\Delta) =$

$t_1R(v_3)+(t_2-t_1)R(v_2)+(t_3-t_2)R(v_1)$. Returning now to other parametric representation of Δ, i.e. $\Delta=[x_0,\sigma,\beta]$ and taking into account (4.1)-(4.3), we obtain $R(\Delta)=x_0R(v_3)+\sigma(R(v_2)-R(v_3))+\beta R(v_1)$. Clearly, this last relation implies $R\in\Omega^{\triangle}$ and the proof is complete. ■

4.5 Characterization of valuable ranking indices on trapezoidal fuzzy numbers

In this section we follow the idea discussed in Section 4.4 on the trapezoidal case. We determine $M_*(\mathbb{F}^T(\mathbb{R}))$ and abstracting of the equivalent orders over $\mathbb{F}^T(\mathbb{R})$, we find all ranking indices $P:\mathbb{F}^T(\mathbb{R})\rightarrow\mathbb{R}$ with the property that $\succeq_P$ satisfies the reasonable properties $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_4$, $\mathbb{A}_4'$, $\mathbb{A}_5$, $\mathbb{A}_5'$ and $\mathbb{A}_6$ on $\mathbb{F}^T(\mathbb{R})$. Then, as it is shown in Section 4.6, we can extend orders defined on $\mathbb{F}^T(\mathbb{R})$ to orders defined on $\mathbb{F}(\mathbb{R})$, so that the basic requirements are preserved. A different idea to rank fuzzy numbers through trapezoidal fuzzy numbers can be found in [25], where fuzzy numbers are ranked using their trapezoidal approximations preserving the ambiguity and value. Actually, the topic of the approximation of fuzzy numbers by fuzzy numbers with simpler form and the topic of the ranking of fuzzy numbers with simpler form are quite related. The aim of both is to simplify as much as possible the way of the information processing that appears as fuzzy numbers.

In this section we use a trapezoidal fuzzy number representation which seems more suitable for obtaining the main results. Namely, we consider the α-cut of a trapezoidal fuzzy number T in the following form (see, e.g., [4])

$$T_\alpha=[x_0-\sigma+\sigma\alpha,y_0+\beta-\beta\alpha],\quad \alpha\in[0,1], \tag{4.21}$$

where $x_0,y_0,\sigma,\beta\in\mathbb{R}$, $\sigma\geq0$, $\beta\geq0$ and $x_0\leq y_0$ and we denote $T=[x_0,y_0,\sigma,\beta]$. Comparing to the classical representation of T (see Section 1.3.4), i.e. $T=(t_1,t_2,t_3,t_4)$, it is immediate that

$$\begin{aligned} t_1&=x_0-\sigma,\\ t_2&=x_0,\\ t_3&=y_0,\\ t_4&=y_0+\beta. \end{aligned}$$

According to (1.72), (1.79) and (1.78), after some simple calculations, we get the expected value, ambiguity and value of a trapezoidal fuzzy number $T=[x_0,y_0,\sigma,\beta]$ as

$$
\begin{aligned}
EV(T) &= \frac{1}{2}x_0 + \frac{1}{2}y_0 - \frac{1}{4}\sigma + \frac{1}{4}\beta, \\
Amb(T) &= -\frac{1}{2}x_0 + \frac{1}{2}y_0 + \frac{1}{6}\sigma + \frac{1}{6}\beta, \\
Val(T) &= \frac{1}{2}x_0 + \frac{1}{2}y_0 - \frac{1}{6}\sigma + \frac{1}{6}\beta.
\end{aligned}
$$

These main characteristics of a trapezoidal fuzzy number $T = [x_0, y_0, \sigma, \beta]$ suggest to consider the following quantity

$$
R(T) = ax_0 + by_0 + c\sigma + d\beta \tag{4.22}
$$

where $a,b,c,d \in \mathbb{R}$ are fixed. The function $R : \mathbb{F}^T(\mathbb{R}) \to \mathbb{R}$ is additive and positive homogenous. Let us introduce the following set

$$
\begin{aligned}
\Omega^T = \big\{ R : \mathbb{F}^T(\mathbb{R}) \to \mathbb{R} \ | \ &\exists a,b,c,d \in \mathbb{R} \text{ such that} \\
&R([x_0, y_0, \sigma, \beta]) = ax_0 + by_0 + c\sigma + d\beta \big\}.
\end{aligned}
$$

It can be shown that $M_*(\mathbb{F}^T(\mathbb{R})) \subseteq \Omega^T$. Because of the relationship between $M_*(\mathbb{F}^T(\mathbb{R}))$ and $M(\mathbb{F}^T(\mathbb{R}))$ it justifies a detailed study of the set Ω^T. For this purpose one may find necessary and sufficient conditions for a,b,c,d such that $R \in \Omega^T$ could be used to rank effectively trapezoidal fuzzy numbers. We have already mentioned in the previous sections that requirements $\mathbb{A}_1$, $\mathbb{A}_2$ and $\mathbb{A}_3$ hold for the ordering $\succeq_R$ on $\mathbb{F}^T(\mathbb{R})$ since $\succeq_R$ is generated by a ranking index. Then, since $R(T+S) = R(T) + R(S)$ for all $R \in \Omega^T$ and $T, S \in \mathbb{F}^T(\mathbb{R})$, it results that properties $\mathbb{A}_5$ and $\mathbb{A}'_5$ hold. Therefore, it remains to find necessary and sufficient conditions such that properties $\mathbb{A}_4$, $\mathbb{A}'_4$, $\mathbb{A}''_4$ and $\mathbb{A}_6$ are satisfied. The proof of the following theorem is similar to that considered for the triangular fuzzy numbers (see Section 4.4 or [36]).

Theorem 4.10. *([36]) Consider $R \in \Omega^T$ such that $R([x_0, y_0, \sigma, \beta]) = ax_0 + by_0 + c\sigma + d\beta$.*

(i) The order $\succeq_R$ satisfies $\mathbb{A}_4$ if and only if $a \geq 0$, $b \geq 0$, $c \leq 0$, $d \geq 0$, $a+b+c \geq 0$ and $a+b-d \geq 0$.
(ii) The order $\succeq_R$ satisfies $\mathbb{A}'_4$ if and only if $a \geq 0$, $b \geq 0$, $a+b > 0$, $c \leq 0$, $d \geq 0$, $a+b+c \geq 0$ and $a+b-d \geq 0$.
(iii) The ranking index R satisfies $\mathbb{A}''_4$ if and only if $a \geq 0$, $b \geq 0$, $a+b = 1$, $c \in [-1,0]$ and $d \in [0,1]$.
(iv) The order $\succeq_R$ satisfies $\mathbb{A}_6$ if and only if $a = b$ and $c+d = 0$.
(v) $R \in M_*(\mathbb{F}^T(\mathbb{R}))$ if and only if $a = b = \frac{1}{2}$, $c \in [-1,0]$ and $c+d = 0$.

According to Theorem 4.10 it is easy to deduce that some already introduced ranking indices satisfy one or more properties $\mathbb{A}_4$, $\mathbb{A}'_4$, $\mathbb{A}''_4$, $\mathbb{A}_5$, $\mathbb{A}'_5$, $\mathbb{A}_6$ or they belong to $M_*(\mathbb{F}^T(\mathbb{R}))$. Consider the following two examples (see [36]).

Example 4.13. Let us consider an index discussed in Example 4.9. If $T = [x_0, y_0, \sigma, \beta]$ is a trapezoidal fuzzy number then by simple calculations (see (4.22) and (4.22)) we get

$$ri(\lambda,\delta)(T) = \frac{\lambda-\delta}{2}x_0 + \frac{\lambda+\delta}{2}y_0 + \frac{\delta-\lambda}{6}\sigma + \frac{\lambda+\delta}{6}\beta$$

and it is immediate that $ri \in \Omega^T$ for any λ and δ. By Theorem 4.10 (ii) we conclude that $\succeq_{ri(\lambda,\delta)}$ satisfies $\mathbb{A}_4'$ for any λ and δ. By Theorem 4.10 (iii) we deduce that $ri(\lambda,\delta)$ satisfies $\mathbb{A}_4''$ if and only if $\lambda = 1$. Moreover, by Theorem 4.10 (iv) we get that $\succeq_{ri(\lambda,\delta)}$ satisfies $\mathbb{A}_6$ if and only if $\delta = 0$. Therefore $ri(\lambda,\delta) \in M(\mathbb{F}^T(\mathbb{R}))$ and hence $\succeq_{ri(\lambda,\delta)}$ satisfies $\mathbb{A}_1, \mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_4$, $\mathbb{A}_4'$, $\mathbb{A}_5$, $\mathbb{A}_5'$ and $\mathbb{A}_6$ if and only if $\delta = 0$. If $\lambda < 1$ then $ri(\lambda,\delta) \in M(\mathbb{F}^T(\mathbb{R})) \setminus M_*(\mathbb{F}^T(\mathbb{R}))$. Moreover, if $\lambda = 1$ then we get $ri(\lambda,\delta) = ri(1,0) = Val \in M_*(\mathbb{F}^T(\mathbb{R}))$ which is the ranking index proposed for the first time in [97]. □

Example 4.14. In [4] the authors have considered the magnitude of a trapezoidal fuzzy number, namely a function $Mag_f : \mathbb{F}^T(\mathbb{R}) \to \mathbb{R}$ defined by

$$Mag_f(T) = \frac{1}{2}\left(\int_0^1 (T_L(\alpha) + T_U(\alpha) + x_0 + y_0) f(\alpha) d\alpha\right),$$

where $T = [x_0, y_0, \sigma, \beta]$ is an arbitrary trapezoidal fuzzy number and f is a non-negative and nondecreasing function on $[0,1]$ such that $f(0) = 0$, $f(1) = 1$ and $\int_0^1 f(\alpha)d\alpha = 1/2$. By simple calculations we get

$$Mag_f(T) = \frac{1}{2}x_0 + \frac{1}{2}y_0 + \frac{\sigma}{2}\int_0^1 f(\alpha)(\alpha-1)d\alpha + \frac{\beta}{2}\int_0^1 f(\alpha)(1-\alpha)d\alpha, \quad (4.23)$$

and hence one can easily conclude (see Theorem 4.10, (v)) that $Mag_f \in M_*(\mathbb{F}^T(\mathbb{R}))$. Note that in [4] the authors dealt with the particular case $f(\alpha) = \alpha$ for which

$$Mag(T) = \frac{1}{2}x_0 + \frac{1}{2}y_0 - \frac{1}{12}\sigma + \frac{1}{12}\beta.$$

□

As in the case of triangular fuzzy numbers it is interesting whether there exists any other ranking index $R \in M_*(\mathbb{F}^T(\mathbb{R}))$ which does not belong to Ω^T. The answer to this question is negative. Hence now we can present the main result of this section.

Theorem 4.11. *([36]) Let us consider a ranking index $R : \mathbb{F}^T(\mathbb{R}) \to \mathbb{R}$. Then $R \in M_*(\mathbb{F}^T(\mathbb{R}))$ if and only if there exists $c \in [-1,0]$ such that for some $T \in \mathbb{F}^T(\mathbb{R})$ of the form $T = [x_0, y_0, \sigma, \beta]$ we have*

$$R(T) = \frac{1}{2}x_0 + \frac{1}{2}y_0 + c\sigma - c\beta. \quad (4.24)$$

Proof. The proof is based on the same idea as in Theorem 4.9. By Theorem 4.10 (v) it is easily seen that we can obtain the desired conclusion by proving that $R \in M_*(\mathbb{F}^T(\mathbb{R}))$ implies $R \in \Omega^T$. Firstly, by Theorem 4.4 (i) we know that R is linear on $\mathbb{F}^T(\mathbb{R})$. We continue with a constructive proof although we could use the Riesz representation theorem for linear functionals.

Let $T \in \mathbb{F}^T(\mathbb{R})$, $T = (t_1, t_2, t_3, t_4)$, denote a trapezoidal fuzzy number in its usual representation. Moreover, let us consider the following trapezoidal fuzzy numbers

$$v_1 = (0,0,0,1),$$
$$v_2 = (0,0,1,1),$$
$$v_3 = (0,1,1,1),$$
$$v_4 = (1,1,1,1).$$

Having in mind the addition and the scalar multiplication of fuzzy numbers we get that $T = t_1 v_4 + (t_2 - t_1)v_3 + (t_3 - t_2)v_2 + (t_4 - t_3)v_1$. The linearity of R implies $R(T) = t_1 R(v_4) + (t_2 - t_1)R(v_3) + (t_3 - t_2)R(v_2) + (t_4 - t_3)R(v_1)$. Returning now to the other parametric representation of T, i.e. $T = [x_0, y_0, \sigma, \beta]$ and taking into account (4.22)-(4.22), we obtain $R(T) = x_0(R(v_4) - R(v_2)) + y_0 R(v_2) + \sigma(R(v_3) - R(v_4)) + \beta R(v_1)$. Clearly, this last relation implies that $R \in \Omega^T$ and the proof is complete. ■

Now we may characterize some classes of ranking indices over $\mathbb{F}^T(\mathbb{R})$ which generate orders satisfying all or just some of the desired properties. Here Theorem 4.1 might be helpful for searching equivalent orders that satisfy requirement $\mathbb{A}_4''$ on $\mathbb{F}^T(\mathbb{R})$.

Corollary 4.6. *([36])*

(i) If $R \in M(\mathbb{F}^T(\mathbb{R}))$ then there exists $R_* \in M_*(\mathbb{F}^T(\mathbb{R}))$ such that $\succeq_R$ and $\succeq_{R_*}$ are equivalent. Moreover, there exists $c \in [-1,0]$ such that $R_*(T) = \frac{1}{2}x_0 + \frac{1}{2}y_0 + c\sigma - c\beta$ for some $T = [x_0, y_0, \sigma, \beta] \in \mathbb{F}^T(\mathbb{R})$.

(ii) If $R : \mathbb{F}^T(\mathbb{R}) \to \mathbb{R}$ is a ranking index such that $\succeq_R$ satisfies $\mathbb{A}_4'$ and $\mathbb{A}_5$ then $\succeq_R$ satisfies also $\mathbb{A}_5'$. Moreover, there exists an additive ranking index $R_+ : \mathbb{F}^T(\mathbb{R}) \to \mathbb{R}$ satisfying $\mathbb{A}_4''$ on $\mathbb{F}^T(\mathbb{R})$ which generates on $\mathbb{F}^T(\mathbb{R})$ an order equivalent to $\succeq_R$.

(iii) If $R : \mathbb{F}^T(\mathbb{R}) \to \mathbb{R}$ is a ranking index such that $\succeq_R$ satisfies $\mathbb{A}_4'$ and $\mathbb{A}_6$ then there exists a scale invariant ranking index $R_\times : \mathbb{F}^T(\mathbb{R}) \to \mathbb{R}$ which satisfies $\mathbb{A}_4''$ on $\mathbb{F}^T(\mathbb{R})$ and generates on $\mathbb{F}^T(\mathbb{R})$ an order equivalent to $\succeq_R$.

Proof. (i) By the proof of Theorem 4.1 we obtain the existence of $R_* : \mathbb{F}^T(\mathbb{R}) \to \mathbb{R}$ which satisfies $\mathbb{A}_4''$ and such that the order $\succeq_{R_*}$ on $\mathbb{F}^T(\mathbb{R})$ is equivalent to $\succeq_R$. This implies that $R_* \in M_*(\mathbb{F}^T(\mathbb{R}))$. Next steps of the proof follow immediately from Theorem 4.11.

(ii) and (iii) The proofs are immediate by substituting $\mathscr{S} = \mathbb{F}^T(\mathbb{R})$ into Corollaries 4.2 and 4.3 respectively. ■

To illustrate the result given in Corollary 4.6 (i) let us consider the following example (see [36]).

Example 4.15. In the very recent paper [172] a new ranking index was proposed. For any trapezoidal fuzzy number $T = [x_0, y_0, \sigma, \beta]$ it becomes

$$Q^{\gamma}_{S-D}([x_0, y_0, \sigma, \beta]) = (1-\gamma)x_0 + (1-\gamma)y_0 - \frac{(1-\gamma)^2}{2}\sigma + \frac{(1-\gamma)^2}{2}\beta,$$

where $\gamma \in [0,1]$ is interpreted as a decision level. By Theorem 4.10, (v) we know that $Q^{\gamma}_{S-D} \in M_*(\mathbb{F}^T(\mathbb{R}))$ if and only if $\gamma = \frac{1}{2}$. It is interesting to note that $Q^{\gamma}_{S-D} \in M(\mathbb{F}^T(\mathbb{R}))$ for any $\gamma < 1$. Therefore, by Theorem 4.1), there exists a ranking index $\left(Q^{\gamma}_{S-D}\right)_*$ defined by

$$\left(Q^{\gamma}_{S-D}\right)_*([x_0, y_0, \sigma, \beta]) = \frac{1}{2}x_0 + \frac{1}{2}y_0 - \frac{1-\gamma}{4}\sigma + \frac{1-\gamma}{4}\beta,$$

such that $\left(Q^{\gamma}_{S-D}\right)_* \in M_*(\mathbb{F}^T(\mathbb{R}))$ and such that the orders generated by $\left(Q^{\gamma}_{S-D}\right)_*$ and Q^{γ}_{S-D} are equivalent. □

4.6 Ranking fuzzy numbers through trapezoidal fuzzy numbers

By Theorem 4.11 we found the class $M_*(\mathbb{F}^T(\mathbb{R}))$ consisting of all ranking indices that generate orders over the space of trapezoidal fuzzy numbers satisfying requirements $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_4''$, $\mathbb{A}_5$, $\mathbb{A}_5'$ and $\mathbb{A}_6$. Below we point out that for any $R \in M_*(\mathbb{F}^T(\mathbb{R}))$ there exists $\overline{R} \in M_*(\mathbb{F}(\mathbb{R}))$ such that $\overline{R}(T) = R(T)$ for any trapezoidal fuzzy number T. It means that $\overline{R}$ is an extension of R on $\mathbb{F}(\mathbb{R})$ so that all desirable properties $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_4''$, $\mathbb{A}_5$, $\mathbb{A}_5'$ and $\mathbb{A}_6$ hold on $\mathbb{F}(\mathbb{R})$.

Theorem 4.12. *([36]) Let us consider an operator* $\mathscr{T} : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^T(\mathbb{R})$ *and* $R \in M_*(\mathbb{F}^T(\mathbb{R}))$. *If* $\mathscr{T}$ *is linear and* $\operatorname{supp}(\mathscr{T}(A)) \subseteq \operatorname{supp}(A)$ *for any* $A \in \mathbb{F}(\mathbb{R})$ *then* $\overline{R} : \mathbb{F}(\mathbb{R}) \to \mathbb{R}$ *given by* $\overline{R}(A) = R(\mathscr{T}(A))$ *is linear and* $\overline{R} \in M_*(\mathbb{F}(\mathbb{R}))$.

Proof. By Theorem 4.4 or Theorem 4.11 we conclude that R is linear on $\mathbb{F}^T(\mathbb{R})$. Thus $\overline{R}$ is linear on $\mathbb{F}(\mathbb{R})$, since $\overline{R}$ is the composition of the linear operators R and $\mathscr{T}$. It remains to prove that requirements $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_4''$, $\mathbb{A}_5$, $\mathbb{A}_5'$ and $\mathbb{A}_6$ hold on $\mathbb{F}(\mathbb{R})$ for $\overline{R}$ and $\succeq_{\overline{R}}$. Properties $\mathbb{A}_1$, $\mathbb{A}_2$ and $\mathbb{A}_3$ are obviously satisfied. Moreover, since $\overline{R}$ is linear, we easily obtain that $\mathbb{A}_5$, $\mathbb{A}_5'$ and $\mathbb{A}_6$ are satisfied too. Now, considering an arbitrary fuzzy number A, since $R \in M_*\left(\mathbb{F}^T(\mathbb{R})\right)$, we get that $R(\mathscr{T}(A)) \in \operatorname{supp}(\mathscr{T}(A))$. Since $\operatorname{supp}(\mathscr{T}(A)) \subseteq \operatorname{supp}(A)$ and $R(\mathscr{T}(A)) = \overline{R}(A)$ we conclude that $\overline{R}(A) \in \operatorname{supp}(A)$ which proves that $\mathbb{A}_4''$ holds for $\overline{R}$ on $\mathbb{F}(\mathbb{R})$ and the theorem is proved. ■

The expected value (1.72) generates an order over $\mathbb{F}(\mathbb{R})$ which satisfies all basic requirements. Nevertheless, if $\int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha$ is difficult to compute for a given fuzzy number A, the ranking based on the expected value is not practical. So it is natural to ask if it is possible to provide orders over $\mathbb{F}(\mathbb{R})$ which are easy to use from computational point of view and which satisfy the basic requirements discussed in Section 4.2. In the following example we propose a method to extend an order from $\mathbb{F}^T(\mathbb{R})$ to $\mathbb{F}(\mathbb{R})$.

Example 4.16. ([36]) If $T = (t_1, t_2, t_3, t_4)$ is a trapezoidal fuzzy number then $EV(T) = \frac{1}{4}(t_1 + t_2 + t_3 + t_4) = \frac{1}{4}(T_L(0) + T_L(1) + T_U(1) + T_U(0))$. Let us define an operator $\mathscr{T} : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^T(\mathbb{R})$ given by

$$\mathscr{T}(A) = (A_L(0), A_L(1), A_U(1), A_U(0)).$$

$\mathscr{T}(A)$ is the unique trapezoidal fuzzy number which preserves the support and the core of $A \in \mathbb{F}(\mathbb{R})$. Let us consider $\overline{R} : \mathbb{F}(\mathbb{R}) \to \mathbb{R}$ given by

$$\overline{R}(A) = \frac{1}{4}(A_L(0) + A_L(1) + A_U(0) + A_U(1))$$

and R which is the restriction of $\overline{R}$ on the subset $\mathbb{F}^T(\mathbb{R})$, i.e. $\overline{R}(T) = R(T) = EV(T)$ for all $T \in \mathbb{F}^T(\mathbb{R})$. It is easy to check that all assumptions in Theorem 4.12 are satisfied for $\overline{R}$, R and $\mathscr{T}$, and therefore $\overline{R} \in M_*(\mathbb{F}(\mathbb{R}))$. □

The above discussed operator $\overline{R}$ is very convenient from the computational point of view and it can be used to compare two fuzzy numbers expressed either in the explicit or parametric way.

It is worth noting that we can extend each order given by the operator $R \in M_*(\mathbb{F}^T(\mathbb{R}))$ to an order over $\mathbb{F}(\mathbb{R})$ which is easy to handle from the computational point of view and satisfies basic requirements considered in Section 4.2.

Theorem 4.13. *([36]) If $R \in M_*(\mathbb{F}^T(\mathbb{R}))$ then there exists $\overline{R} : \mathbb{F}(\mathbb{R}) \to \mathbb{R}$ such that $\overline{R} \in M_*(\mathbb{F}(\mathbb{R}))$. Moreover, $\overline{R}(T) = R(T)$ for all $T \in \mathbb{F}^T(\mathbb{R})$.*

Proof. If $R \in M_*(\mathbb{F}^T(\mathbb{R}))$ then there exists $c \in [-1, 0]$ such that

$$R(T) = \frac{1}{2}x_0 + \frac{1}{2}y_0 + c\sigma - c\beta$$

for $T = [x_0, y_0, \sigma, \beta]$ (see Theorem 4.11). Since $x_0 = T_L(1), \sigma = T_L(1) - T_L(0), y_0 = T_U(1), \beta = T_U(0) - T_U(1)$, we get

$$R(T) = (\frac{1}{2} + c)(T_L(1) + T_U(1)) - c(T_L(0) + T_U(0)).$$

Define $\overline{R} : \mathbb{F}(\mathbb{R}) \to \mathbb{R}$ by

$$\overline{R}(A) = (\frac{1}{2} + c)(A_L(1) + A_U(1)) - c\,(A_L(0) + A_U(0))$$

and $\mathscr{T} : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^T(\mathbb{R})$ such that

$$\mathscr{T}(A) = [A_L(1), A_U(1), A_L(1) - A_L(0), A_U(0) - A_U(1)]\,.$$

It is obvious that $\overline{R}$, R and $\mathscr{T}$ satisfy the assumptions in Theorem 4.12 and hence the conclusion is immediate. ■

In the end of this section we give an explicit form of such operator $\overline{R} : \mathbb{F}(\mathbb{R}) \to \mathbb{R}$ that $\overline{R} \in M_*(\mathbb{F}(\mathbb{R}))$ and such that for any fuzzy numbers A and B with identical support and core we have $A \sim_{\overline{R}} B$.

Theorem 4.14. *([36]) Suppose $\overline{R} \in M_*(\mathbb{F}(\mathbb{R}))$ is such that for any fuzzy numbers A and B with identical support and core we have $A \sim_{\overline{R}} B$. Then there exists $c \in [-1, 0]$ such that*

$$\overline{R}(A) = (\frac{1}{2} + c)(A_L(1) + A_U(1)) - c(A_L(0) + A_U(0)).$$

Proof. By Theorem 4.11 there exist such $c \in [-1, 0]$ that $\overline{R}(T) = (\frac{1}{2} + c)(T_L(1) + T_U(1)) - c(T_L(0) + T_U(0))$ for all $T \in \mathbb{F}^T(\mathbb{R})$. Now, for an arbitrary fuzzy number A we consider the following trapezoidal fuzzy number

$$T_A = [A_L(1), A_U(1), A_L(1) - A_L(0), A_U(0) - A_U(1)]\,,$$

i.e. A and T_A have the same support and core. By the assumptions $\overline{R}(A) = \overline{R}(T_A)$ and hence

$$\begin{aligned}\overline{R}(T_A) &= (\frac{1}{2} + c)((T_A)_L(1) + (T_A)_U(1)) - c((T_A)_L(0) + (T_A)_U(0)) \\ &= (\frac{1}{2} + c)(A_L(1) + A_U(1)) - c(A_L(0) + A_U(0)),\end{aligned}$$

we obtain the desired conclusion. ■

Problems

4.1. Prove that the ranking index $P : \mathbb{F}^{\Delta}(\mathbb{R}) \to \mathbb{R}$ given by

$$P((a,b,c)) = \frac{(a+b+c)\,(a+4b+c)}{9\,(a+2b+c)},$$

(see [69]) satisfies $\mathbb{A}_4'$, while $R : \mathbb{F}^{\Delta}(\mathbb{R}) \to \mathbb{R}$ such that

$$R((a,b,c)) = 2P((a,b,c))$$

satisfies $\mathbb{A}_4''$.

4.2. Prove that the ranking index $P : \mathbb{F}(\mathbb{R}) \to \mathbb{R}$ defined as (see [67])

$$P(A) = \frac{1}{2(M-m)} \left(\int_0^1 A_L(\alpha) d\alpha + \int_0^1 A_U(\alpha) d\alpha - 2m \right),$$

where $m, M \in \mathbb{R}$ and $M \neq m$ satisfies $\mathbb{A}_4'$.

4.3. Consider a set $\mathscr{S} \subseteq \mathbb{F}(\mathbb{R})$ and suppose that there exists a function $f : \mathscr{S} \to \mathbb{F}^T(\mathbb{R})$ such that $\text{supp}(f(A)) \subseteq \text{supp}(A)$ for all $A \in \mathscr{S}$. If $R : \mathbb{F}^T(\mathbb{R}) \to \mathbb{R}$ is a ranking index such that $\succeq_R$ satisfies $\mathbb{A}_4$ (or $\mathbb{A}_4'$) on $\mathbb{F}^T(\mathbb{R})$ then $\overline{R} : \mathscr{S} \to \mathbb{R}$ defined as $\overline{R} = R \circ f$ is a ranking index such that $\succeq_{\overline{R}}$ satisfies $\mathbb{A}_4$ (or $\mathbb{A}_4'$) on $\mathscr{S}$.

4.4. Consider a set $\mathscr{S} \subseteq \mathbb{F}(\mathbb{R})$ such that $\mathscr{S} + \mathscr{S} \subseteq \mathscr{S}$ and suppose that there exists an additive function $f : \mathscr{S} \to \mathbb{F}^T(\mathbb{R})$ such that $\text{supp}(f(A)) \subseteq \text{supp}(A)$ for all $A \in \mathscr{S}$. If $R : \mathbb{F}^T(\mathbb{R}) \to \mathbb{R}$ is a defuzzifier such that $\succeq_R$ satisfies $\mathbb{A}_5$ and $\mathbb{A}_5'$ on $\mathbb{F}^T(\mathbb{R})$ then $\overline{R} : \mathscr{S} \to \mathbb{R}$ defined as $\overline{R} = R \circ f$ is a defuzzifier such that $\succeq_{\overline{R}}$ satisfies $\mathbb{A}_5$ and $\mathbb{A}_5'$ on $\mathscr{S}$.

4.5. Consider a set $\mathscr{S} \subseteq \mathbb{F}(\mathbb{R})$ such that $\lambda \cdot \mathscr{S} \subseteq \mathscr{S}$ for all $\lambda \in \mathbb{R}$ and suppose that there exists a homogenous function $f : \mathscr{S} \to \mathbb{F}^T(\mathbb{R})$ such that $\text{supp}(f(A)) \subseteq \text{supp}(A)$ for all $A \in \mathscr{S}$. If $R : \mathbb{F}^T(\mathbb{R}) \to \mathbb{R}$ is a defuzzifier such that $\succeq_R$ satisfies $\mathbb{A}_6$ on $\mathbb{F}^T(\mathbb{R})$ then $\overline{R} : \mathscr{S} \to \mathbb{R}$ defined as $\overline{R} = R \circ f$ is a defuzzifier such that $\succeq_{\overline{R}}$ satisfies $\mathbb{A}_6$ on $\mathscr{S}$.

4.6. Consider an arbitrary set of all $L-L$ fuzzy numbers for some given L (see Definition 1.10). Consider a ranking index $R : \mathbb{F}_{L,L}(\mathbb{R}) \to \mathbb{R}$ such that $R \in M(\mathbb{F}_{L,L}(\mathbb{R}))$. Prove that there exists $R_* \in M_*(\mathbb{F}_{L,L}(\mathbb{R}))$ such that $\succeq_R$ and $\succeq_{R_*}$ are equivalent. Moreover, prove that there exists $c \in [-1, 0]$ such that for some $A = [x, y, \sigma, \beta]_{L,L} \in \mathbb{F}_{L,L}(\mathbb{R})$, where $A_\alpha = [x - \sigma + \sigma L^{-1}(\alpha), y + \beta - \beta R^{-1}(\alpha)]$, $\alpha \in [0, 1]$, we have $R_*(A) = \frac{1}{2}x + \frac{1}{2}y + c\sigma - c\beta$. (Here we use the same type of the left-spread right-spread representation as we did in Section 4.5 for the case of the trapezoidal fuzzy numbers. We also observe that this result extends Corollary 4.6 to the more general case of $L-L$ fuzzy numbers.)

4.7. Let $P : \mathbb{F}(\mathbb{R}) \to \mathbb{R}$ be a ranking index given by $P(A) = (A_L(0))^3$. Prove that P does not satisfy $\mathbb{A}_4''$ and find a ranking index $R : \mathbb{F}(\mathbb{R}) \to \mathbb{R}$ which satisfies $\mathbb{A}_4''$ such that $\succeq_R$ and $\succeq_P$ are equivalent.

4.8. Prove that operator $EV : \mathbb{F}^T(\mathbb{R}) \to \mathbb{R}$ which attributes for any trapezoidal fuzzy number $T = [x_0, y_0, \sigma, \beta]$ its expected value, i.e. $EV(T) = \frac{1}{2}x_0 + \frac{1}{2}y_0 - \frac{1}{4}\sigma + \frac{1}{4}\beta$, is an element of $M_*(\mathbb{F}^T(\mathbb{R}))$.

4.9. Let $R : \mathbb{F}^T(\mathbb{R}) \to \mathbb{R}$ by defined as (see [148])

$$R([x_0, y_0, \sigma, \beta]) = (1-\gamma)x_0 + \gamma y_0 + \frac{\gamma - 1}{2}\sigma + \frac{\gamma}{2}\beta,$$

where $\gamma \in [0, 1]$ represents the degree of optimism of a decision maker. Find a necessary and sufficient condition so that R satisfies $\mathbb{A}_6$.

4.10. Let $EV : \mathbb{F}(\mathbb{R}) \to \mathbb{R}$ denote the operator which attributes to a fuzzy number its expected value (1.72). Applying Theorem 4.12 prove that $EV \in M_*(\mathbb{F}(\mathbb{R}))$.

4.11. Let $M_{\underline{f},\overline{f}} : \mathbb{F}^T(\mathbb{R}) \to \mathbb{R}$ be defined by (see [171])

$$M_{\underline{f},\overline{f}}([x_0,y_0,\sigma,\beta]) = \frac{1}{2}x_0 + \frac{1}{2}y_0 + \frac{\sigma}{2}\int_0^1 \underline{f}(\alpha)(\alpha-1)d\alpha + \frac{\beta}{2}\int_0^1 \overline{f}(\alpha)(1-\alpha)d\alpha,$$

where the weighting functions $\underline{f},\overline{f} : [0,1] \to \mathbb{R}$ are non-negative, increasing and such that $\int_0^1 \underline{f}(\alpha)d\alpha = \int_0^1 \overline{f}(\alpha)d\alpha = 1$. Prove that $M_{\underline{f},\overline{f}} \in M_*(\mathbb{F}^T(\mathbb{R}))$.

4.12. Let $f : [0,1] \to [0,\infty)$ be a strictly monotone function and suppose the ranking index $R : \mathbb{F}^T(\mathbb{R}) \to \mathbb{R}$ is defined by (see [84])

$$R([x_0,y_0,\sigma,\beta]) = \frac{2-\omega}{2}x_0 + \frac{2-\omega}{2}y_0 - \frac{1-\omega}{2}\sigma + \frac{1-\omega}{2}\beta,$$

where

$$\omega = \frac{\int_0^1 \alpha f(\alpha)d\alpha}{\int_0^1 f(\alpha)d\alpha}.$$

(i) Prove that $R \in M(\mathbb{F}^T(\mathbb{R}))$ and $R \notin M_*(\mathbb{F}^T(\mathbb{R}))$.
(ii) Prove that $R_* \in M_*(\mathbb{F}^T(\mathbb{R}))$ and $\succeq_{R_*}$ is equivalent to $\succeq_R$, where $R_* = \frac{1}{2-\omega}R$.

4.13. Find all possible defuzzifiers in the set $M_*\left(\mathbb{F}^T(\mathbb{R})\right)$ which generate orders that satisfy the following property of an order $\succeq$: If $A = (t_1,t_2,t_3,t_4)$ and $B = (s_1,s_2,s_3,s_4)$ such that $t_i \geq s_i$ for every $i \in 1,2,3,4$, then $A \succeq B$.

4.14. Suppose $\mathscr{S} \subseteq \mathbb{F}(\mathbb{R})$ is such that $\mathbb{R} \subseteq \mathscr{S}$ and let $R : \mathscr{S} \to \mathbb{R}$ be continuous on the restriction of $\mathscr{S}$ on $\mathbb{R}$. Then we know that if $\succeq_R$ satisfies $\mathbb{A}_4'$ on $\mathscr{S}$ then $\succeq_R$ satisfies $\mathbb{A}_4$ on $\mathscr{S}$. Find a set $\mathscr{S} \subseteq \mathbb{F}(\mathbb{R})$ and an order $\succeq$ on $\mathscr{S}$ (generated or not by a ranking index) such that $\succeq$ satisfies $\mathbb{A}_4'$ on $\mathscr{S}$ but it does not satisfy (in general) $\mathbb{A}_4$ on $\mathscr{S}$.

4.15. Prove that the conclusion of Theorem 4.1 also holds if instead of $\mathbb{R} \subseteq \mathscr{S}$ we assume that $\mathscr{S} \cap \mathbb{R}$ is a closed interval in $\mathbb{R}$.

4.16. Prove that all the conclusion in Theorem 4.2 also hold if instead of $\mathbb{R} \subseteq \mathscr{S}$ we assume that $\mathscr{S} \cap \mathbb{R} = [0,\infty)$.

4.17. In [3] the authors propose to rank fuzzy numbers by using L^p-type distances. In [6] the same approach is proposed with a small modification. Namely, let us fix a real $p \geq 1$ and consider an arbitrary fuzzy number A. If $\int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha > 0$ then we take

$$\delta_p(A) = \left(\int_0^1 |A_L(\alpha)|^p\, d\alpha + \int_0^1 |A_U(\alpha)|^p\, d\alpha\right)^{1/p}.$$

If $\int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha < 0$ then

$$\delta_p(A) = -\left(\int_0^1 |A_L(\alpha)|^p \, d\alpha + \int_0^1 |A_U(\alpha)|^p \, d\alpha\right)^{1/p},$$

and finally, if $\int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha = 0$ then $\delta_p(A) = 0$. Now, considering δ_p as a ranking index on $\mathbb{F}^T(\mathbb{R})$, we obtain on $\mathbb{F}^T(\mathbb{R})$ the order $\succeq_{\delta_p}$ generated by δ_p. Prove that for this order neither $\mathbb{A}_4$ nor $\mathbb{A}'_4$ hold in general on $\mathbb{F}^T(\mathbb{R})$.

4.18. Denote by $\mathbb{I}(\mathbb{R})$ the set of all compact intervals in $\mathbb{R}$. Find all the ranking indices (up to equivalent orders) on $\mathbb{I}(\mathbb{R})$ which generate orders satisfying reasonable properties $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_4$, $\mathbb{A}'_4$, $\mathbb{A}_5$, $\mathbb{A}'_5$ and $\mathbb{A}_6$ on $\mathbb{I}(\mathbb{R})$.

4.19. Denote by $\mathbb{I}_+(\mathbb{R})$ the set of all positive compact intervals in $\mathbb{R}$. Find all the ranking indices (up to equivalent orders) on $\mathbb{I}(\mathbb{R})$ which generate orders satisfying reasonable properties $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_4$, $\mathbb{A}'_4$, $\mathbb{A}_5$, $\mathbb{A}'_5$ and $\mathbb{A}_6$ on $\mathbb{I}_+(\mathbb{R})$.

4.20. Suppose $R \in M_*(\mathbb{F}^T(\mathbb{R}))$ and let T be a symmetric trapezoidal fuzzy number with respect to the origin, i.e. $T = [-x_0, x_0, \sigma, \sigma]$. Prove that $R(T) = 0$.

4.21. Suppose $R \in M_*(\mathbb{F}^T(\mathbb{R}))$ and let T_1 and T_2 denote the following trapezoidal fuzzy numbers: $T_1 = [x_0, y_0, \sigma, \sigma]$ and $T_2 = [x_0, y_0, \beta, \beta]$. Prove that $R(T_1) = R(T_2)$.

Chapter 5
Applications

5.1 Approximation of fuzzy numbers and aggregation

In decision making, science and all other activities where data are processed and utilized for making conclusions, the problem of data aggregation turns out important immediately. To make a decision or to draw a final conclusion one often needs a kind of synthesis of all essential information delivered in available data. It is usually achieved by using an appropriate aggregation operator which, roughly speaking, combines several values (numerical or non-numerical) into a single one. Although it can be done in many ways, the desired final result of aggregation should be somehow representative to the initial data set. Thus, an adequate aggregation operator should possess some properties that can be requested for aggregation. These properties can be divided into "formal", which guarantee sound mathematical structures and "behavioral", connected with usefulness in particular practical problems. As the latter are strictly related to specific applications, the former are intensively studied within the framework of the theory of aggregation operators (or aggregation functions) which is nowadays a rapidly developing mathematical domain. For the recent state of art monograph we refer the reader to [99], while for the book more oriented towards practitioners we refer to [44].

However, this section is addressed neither to "formal" nor to "behavioral" properties of aggregation operators but to problems related to data representation that appear in the context of aggregation of fuzzy data. As we have discussed it in Chapter 3 such data may cause serious difficulties in all aspects of fuzzy data processing, including calculus and interpretation, especially if membership functions are too complicated. That is also the reason why practitioners are so interested in various simplification methods applicable to fuzzy inputs and realized through defuzzification or different kinds of approximations, described broadly in Chapter 3.

Whatever approximation method is preferred, one fundamental question related to aggregation remains: When the approximation should be performed: before or after aggregation? In other words, one has to decide whether it is better to simplify initial data before using an aggregation operator or conversely, to aggregate original fuzzy values and then to simplify the output.

Obviously, that there is no unique answer for given question since the diversity of possible situations is too broad to find out a solution that always holds. Anyway, in some cases we may obtain quite interesting solutions and two of such cases are discussed below. The main idea can be found in [37] too.

Suppose now we have a sequence $A_1, \ldots, A_n$ of fuzzy numbers that should be aggregated efficiently. By the word *efficiently* we mean that the output is both representative to the whole set of inputs and its form is simple. Having in mind all the discussions related to trapezoidal approximation and its benefits, given in Section 3.5, let us try to find such trapezoidal fuzzy number $T_{A_1,\ldots,A_n}$ which is the nearest one to all members of the initial data set $A_1, \ldots, A_n$ with respect to the Euclidean distance (1.40). In other words, we are looking for such trapezoidal fuzzy number $T_{A_1,\ldots,A_n}$ which minimizes the distance $D((A_1, \ldots, A_n), T_{A_1,\ldots,A_n})$, where

$$D^2\left((A_1, \ldots, A_n), T_{A_1,\ldots,A_n}\right) = \sum_{i=1}^{n} d^2\left(A_i, T_{A_1,\ldots,A_n}\right). \tag{5.1}$$

Now let us consider the arithmetic mean of given fuzzy numbers $A_1, \ldots, A_n$, denoted by $\overline{A}$, i.e.

$$\overline{A} = \frac{1}{n} \cdot (A_1 + \ldots + A_n), \tag{5.2}$$

where the average is computed according to the Minkowski rules, i.e. (1.21) and (1.29). As it is known the average is a very popular aggregation operator having many interesting properties (see, e.g., [44]). This is the main reason that the average is so often used in practise. Suppose now we want to find the nearest trapezoidal fuzzy number $T(\overline{A})$ of fuzzy number $\overline{A}$ with respect to the distance (1.40) by minimizing $d^2(\overline{A}, T(\overline{A}))$.

It appears that the following theorem holds.

Theorem 5.1. *The trapezoidal fuzzy number nearest to fuzzy numbers $A_1, \ldots, A_n$ is the trapezoidal fuzzy number nearest to fuzzy number $\overline{A}$, i.e.*

$$T_{A_1,\ldots,A_n} = T(\overline{A}).$$

For the proof we refer the reader to [37] (actually, Theorem 5.1 is a direct conclusion of Theorem 6 proved in [37] for the weighted distance (1.42)).

Thus, turning back to the question posed at the beginning of this section: *Is it better to simplify initial data before using an aggregation operator or conversely, to aggregate original fuzzy values and then to simplify the output?* we conclude by Theorem 5.1 that the problem disappears if the average is chosen as the aggregation operator. Actually, Theorem 5.1 shows that the trapezoidal fuzzy number nearest to given family of fuzzy numbers is equivalent to the trapezoidal fuzzy number nearest to their average (provided any solution exists). In other words, there is no difference whether the approximation is performed before or after aggregation.

The following corollary could be also useful for practical calculations.

Corollary 5.1. *The trapezoidal fuzzy number nearest to trapezoidal fuzzy numbers* $A_1,\ldots,A_n$*, where* $A_i = (a_1^i, a_2^i, a_3^i, a_4^i)$*,* $i = 1,\ldots n$*, is*

$$T_{A_1,\ldots,A_n} = \overline{A} = \left(\frac{1}{n}\sum_{i=1}^{n} a_1^i, \frac{1}{n}\sum_{i=1}^{n} a_2^i, \frac{1}{n}\sum_{i=1}^{n} a_3^i, \frac{1}{n}\sum_{i=1}^{n} a_4^i \right). \tag{5.3}$$

Moreover, appreciating the role of the expected interval invariance during approximation, considered broadly in Section 3.5.1, let us assume that the desired fuzzy number $T^*_{A_1,\ldots,A_n}$ which is the nearest one to all members of the initial data set $A_1,\ldots,A_n$ with respect to the Euclidean distance (1.40), should also preserve somehow the expected interval. However, before undertaking any optimization task the last requirement has to be specified more formally. Hence let us assume that the expected interval of a set of fuzzy numbers $A_1,\ldots,A_n$ would be in a natural way defined by aggregating them using average, i.e.

$$\widetilde{EI}(A_1,\ldots,A_n) = \frac{1}{n}\left(EI(A_1) + \ldots + EI(A_n)\right), \tag{5.4}$$

where addition and scalar multiplication are computed according to the Minkowski rules (see (1.21) and (1.29)).

Therefore, our goal now is to find such trapezoidal fuzzy number $T^*_{A_1,\ldots,A_n}$ which minimizes the distance $D((A_1,\ldots,A_n), T^*_{A_1,\ldots,A_n})$, defined by (5.1), and such that

$$EI(T^*_{A_1,\ldots,A_n}) = \widetilde{EI}(A_1,\ldots,A_n). \tag{5.5}$$

It could be shown that the following theorem, similar to Theorem 5.1, holds.

Theorem 5.2. *The trapezoidal fuzzy number nearest to fuzzy numbers* $A_1,\ldots,A_n$ *which preserves the expected interval of* $A_1,\ldots,A_n$ *is the trapezoidal fuzzy number nearest to fuzzy number* $\overline{A}$ *which preserves the weighted expected interval of* $\overline{A}$*, i.e.*

$$T^*_{A_1,\ldots,A_n} = T^*(\overline{A}).$$

For the proof we refer the reader again to [37], because Theorem 5.2 is a direct conclusion of Theorem 10 proved in [37] for the weighted distance (1.42) and the invariance of the weighted expected interval (1.84).

Theorem 5.2 shows that there is no difference whether the approximation preserving the weighted expected interval is performed before or after aggregation with respect to average. Thus a general conclusion from Theorem 5.2 is similar to that obtained for the approximation without condition concerning the expected interval invariance. This remark, of course, does not mean that in both cases we obtain the same outputs. Let us consider the following example.

Example 5.1. Suppose A_1, A_2 and A_3 are fuzzy numbers given by their α-cuts $(A_1)_\alpha = [-1+\alpha^2, 4-2\alpha^2]$, $(A_2)_\alpha = [1+\alpha^2, 3-\alpha^2]$ and $(A_3)_\alpha = [45\sqrt{\alpha}, 46-\sqrt{\alpha}]$, respectively, where $\alpha \in [0,1]$. Then

$$\left(\frac{1}{3}(A_1+A_2+A_3)\right)_\alpha = \left[\frac{2}{3}\alpha^2+15\sqrt{\alpha}, \frac{53}{3}-\alpha^2-\frac{1}{3}\sqrt{\alpha}\right], \quad \alpha \in [0,1].$$

According to Algorithm 1, Step 1 in [195], the trapezoidal fuzzy number nearest to $\frac{1}{3}(A_1+A_2+A_3)$ is

$$T\left(\frac{1}{3}(A_1+A_2+A_3)\right) = \left(\frac{469}{120}, \frac{991}{60}, \frac{991}{60}, \frac{709}{40}\right),$$

and by Theorem 5.1 we get that $\left(\frac{469}{120}, \frac{991}{60}, \frac{991}{60}, \frac{709}{40}\right)$ is also the trapezoidal fuzzy number nearest to A_1, A_2 and A_3.

By Theorem 3.8 one can also find that the trapezoidal fuzzy number nearest to $\frac{1}{3}(A_1+A_2+A_3)$ preserving the expected interval of $\frac{1}{3}(A_1+A_2+A_3)$ is

$$T^*\left(\frac{1}{3}(A_1+A_2+A_3)\right) = \left(\frac{707}{180}, \frac{991}{60}, \frac{991}{60}, \frac{3187}{180}\right).$$

By Theorem 5.2 we conclude immediately that $\left(\frac{707}{180}, \frac{991}{60}, \frac{991}{60}, \frac{3187}{180}\right)$ is the trapezoidal fuzzy number nearest to A_1, A_2 and A_3 which preserves the expected interval of the set of fuzzy numbers A_1, A_2 and A_3. □

Although in general $T_{A_1,\dots,A_n}$ and $T^*_{A_1,\dots,A_n}$ may differ, one may indicate some situation where they coincide.

Corollary 5.2. *The trapezoidal fuzzy number nearest to trapezoidal fuzzy numbers* $A_1,\dots,A_n$*, where* $A_i = (a_1^i, a_2^i, a_3^i, a_4^i)$*,* $i = 1,\dots n$*, which preserves the expected interval of* $A_1,\dots,A_n$ *is*

$$T^*_{A_1,\dots,A_n} = \overline{A} = \left(\frac{1}{n}\sum_{i=1}^n a_1^i, \frac{1}{n}\sum_{i=1}^n a_2^i, \frac{1}{n}\sum_{i=1}^n a_3^i, \frac{1}{n}\sum_{i=1}^n a_4^i\right) = T_{A_1,\dots,A_n}. \tag{5.6}$$

Proof. Since the expected interval (1.71) is linear and then taking into account (5.4) we obtain

$$\widetilde{EI}(A_1,\dots,A_n) = \frac{1}{n}(EI(A_1)+\dots+EI(A_n)) = EI\left(\frac{1}{n}(A_1+\dots+A_n)\right) = EI\left(\overline{A}\right).$$

Moreover, having in mind that the trapezoidal approximation operator satisfies the identity criterion (see Definition 3.5), so that the trapezoidal approximation of a trapezoidal fuzzy number is equivalent to that number, the proof is complete. ■

It is worth noting that our results might be also applied to calculate the trapezoidal approximations of the interval-valued fuzzy numbers (intuitionistic fuzzy numbers), discussed in Section 2.1. In the simplest case we may consider a single interval-valued fuzzy number described by the upper and lower membership functions A^+ and A^-, respectively. Thus, according to Theorem 5.1 applied for $n = 2$, we obtain the same trapezoidal approximation whatever we do first: approximate separately A^+ and A^- and then aggregate the solutions or conversely, aggregate A^+ and A^- and then find the desired approximation.

5.2 Fuzzy median of a fuzzy sample

5.2.1 Fuzzy sample median

In descriptive statistics we can distinguish three groups of the so-called summary statistics to characterize briefly a data set under study: measures of location, measures of dispersion and measures of shape. The *central tendency measures*, as a subset of location measures, seem to be the most important because they characterize values typical for a sample. The most popular central tendency measure is the *arithmetic mean*. As an estimator of the population mean it possesses many desired properties but also a strong drawback: it is sensitive to outliers. A *sample median* or a median in brief, is a central tendency measure which avoids this unpleasant behavior. The median is usually defined as a middle element in the ordered sequence of observations (if a sample size is odd) or as the average of the two middle observations (if a sample size is even). Therefore, just by its definition the median is not influenced by the extreme values and hence is robust to outliers.

More precisely, if $X_1, \ldots, X_n$ denote a usual crisp sample of real numbers then the median $Med = Med(X_1, \ldots, X_n)$ is defined as

$$Med = \begin{cases} X_{\left(\frac{n+1}{2}\right)} & \text{if} \quad n \text{ is odd} \\ \frac{1}{2}\left(X_{\left(\frac{n}{2}\right)} + X_{\left(\frac{n}{2}+1\right)}\right) & \text{if} \quad n \text{ is even,} \end{cases}$$

where $X_{(1)} \leq \ldots \leq X_{(n)}$ denote order statistics from the sample $X_1, \ldots, X_n$.

Having a fuzzy sample that describes imprecise observations we are also interested in characterizing the most important properties of the data set in a concise way. It appears that a fuzzy counterpart of the arithmetic mean is also sensitive to outliers. Thus it seems naturally to expect that a *fuzzy median* might be a good candidate for a central tendency measure which is robust to outliers in a fuzzy environment. However, when we move to fuzzy data and want to generalize the concept of the median to a fuzzy context, we see immediately that it is not possible to make it straightforwardly since fuzzy numbers do not form a linear order and so we cannot find the “middle” one. Although there are many methods of ranking fuzzy numbers, there is too much subjectivity in them and hence no ranking method is generally accepted and broadly applied. Nevertheless, the idea of a median in a fuzzy environment was not abandoned. A fuzzy generalization of the median may be defined in several ways (see, e.g., [38, 73, 79, 102, 173, 174, 175]) and hence neither its existence nor properties are obvious.

Now let $A_1,\ldots,A_n$ denote a fuzzy sample of imprecise observations, where each A_i is a fuzzy number. A fuzzy sample median $\widetilde{Med} = \widetilde{Med}(A_1,\ldots,A_n)$ of the fuzzy sample $A_1,\ldots,A_n$ was defined in [102] as a fuzzy number with α-cuts $\widetilde{Med}_\alpha(A_1,\ldots,A_n) = [\widetilde{Med}^L_\alpha, \widetilde{Med}^U_\alpha]$ given by

$$\widetilde{Med}^L_\alpha = \begin{cases} (A^L_\alpha)_{(\frac{n+1}{2})} & \text{if} \quad n \text{ is odd} \\ \frac{\left((A^L_\alpha)_{(\frac{n}{2})} + (A^L_\alpha)_{(\frac{n}{2}+1)}\right)}{2} & \text{if} \quad n \text{ is even,} \end{cases} \tag{5.7}$$

$$\widetilde{Med}^U_\alpha = \begin{cases} (A^U_\alpha)_{(\frac{n+1}{2})} & \text{if} \quad \text{ is odd} \\ \frac{\left((A^U_\alpha)_{(\frac{n}{2})} + (A^U_\alpha)_{(\frac{n}{2}+1)}\right)}{2} & \text{if} \quad n \text{ is even,} \end{cases} \tag{5.8}$$

where $(A^L_\alpha)_{(k)}$ denotes the k-th order statistic of a sample $(A_1)_L(\alpha),\ldots,(A_n)_L(\alpha)$ while $(A^U_\alpha)_{(k)}$ is the k-th order statistic of a sample $(A_1)_U(\alpha),\ldots,(A_n)_U(\alpha)$. It can be shown that the fuzzy sample median $\widetilde{Med}$ becomes a traditional crisp sample median if the observations are not vague but crisp.

It was shown in [102] that the fuzzy sample median defined by (5.7) and (5.8) is a consistent estimator of a fuzzy population median. Moreover, some applications of so defined fuzzy sample median were given there.

Unfortunately, the fuzzy sample median defined by (5.7) and (5.8) is difficult to handle and the resulting output because of its complicated shape may not have a straightforward interpretation. Although the family of fuzzy numbers $\mathbb{F}(\mathbb{R})$ consists of objects with diverse membership functions, fuzzy numbers with simpler membership functions are often preferred in practice. This is why another construction of a fuzzy median, inspired by its interpretation in classical probability theory, was proposed in [38]. We describe this idea in the next section.

5.2.2 A fixed-shape fuzzy median

It is clear that the expected value or the median of a sample can be obtained by minimizing certain functions. For example, a probabilistic median of a sample $X_1,\ldots,X_n$ is actually a real value m which minimizes $\sum_{i=1}^n |X_i - y|$, i.e.

$$m = \arg\min_{y\in\mathbb{R}} \sum_{i=1}^n |X_i - y| \,. \tag{5.9}$$

We will adapt the above formula to construct a median of a fuzzy sample.

Suppose that $A_1,\ldots,A_n$ is a sample of fuzzy numbers and let d denote a distance between fuzzy numbers. Inspired by (5.9) let as define a *fuzzy median* as such fuzzy number M which minimizes the distance d between this fuzzy number and the sample under study, i.e.

$$M = \arg \min_{B \in \mathbb{F}(\mathbb{R})} \sum_{i=1}^{n} d(B, A_i). \tag{5.10}$$

As an important remark let us notice that in the classical case the median of a sample can be formulated using the natural ordering between observations. Unfortunately, we do not have the same interpretation using an ordering between fuzzy numbers. This is why further on we will stick to the distance minimization problem.

It may happen that as a solution of the minimization problem (5.10) we obtain a fuzzy number having too complex membership function and hence making it useless in applications. Therefore, we need to relax the above minimization problem by imposing regularity properties to the solution. Here we find a similar motivation as in the case of the approximation of fuzzy numbers broadly discussed in Chapter 3. All these facts make us believe that we should look for a useful fuzzy median among fuzzy numbers with simple membership functions such as trapezoidal, triangular and so on. This is also a motivation for considering the following four definitions (see [38]).

Definition 5.1. A *trapezoidal median* of a fuzzy sample $A_1, \ldots, A_n$ is such a trapezoidal fuzzy number $M^T \in \mathbb{F}^T(\mathbb{R})$ which is closest to the fuzzy sample $A_1, \ldots, A_n$ with respect to distance d, i.e.

$$M^T = \arg \min_{B \in \mathbb{F}^T(\mathbb{R})} \sum_{i=1}^{n} d(B, A_i). \tag{5.11}$$

Definition 5.2. A *triangular median* of a fuzzy sample $A_1, \ldots, A_n$ is such a triangular fuzzy number $M^{\triangle} \in \mathbb{F}^{\Delta}(\mathbb{R})$ which is closest to the fuzzy sample $A_1, \ldots, A_n$ with respect to distance d, i.e.

$$M^{\triangle} = \arg \min_{B \in \mathbb{F}^{\Delta}(\mathbb{R})} \sum_{i=1}^{n} d(B, A_i). \tag{5.12}$$

Definition 5.3. An *interval median* of a fuzzy sample $A_1, \ldots, A_n$ is such an interval $M^I \in \mathbb{I}$ which is closest to the fuzzy sample $A_1, \ldots, A_n$ with respect to distance d, i.e.

$$M^I = \arg \min_{B \in \mathbb{I}} \sum_{i=1}^{n} d(B, A_i). \tag{5.13}$$

Definition 5.4. A *crisp median* of a fuzzy sample $A_1, \ldots, A_n$ is such a real number $M^C \in \mathbb{R}$ which is closest to the fuzzy sample $A_1, \ldots, A_n$ with respect to distance d, i.e.

$$M^C = \arg \min_{B \in \mathbb{R}} \sum_{i=1}^{n} d(B, A_i). \tag{5.14}$$

Of course, we have here a kind of hierarchy, since each triangular fuzzy number is a trapezoidal fuzzy number and so on, but it seems that all four medians suggested above may be interesting in practice.

Depending on a certain application one can use any of the metrics listed by us in Section 1.5. As it will be revealed in the next section, existence results for the fuzzy medians proposed in Definition 5.1-5.4 will be provided for broad classes of metrics including those from Section 1.5.

Note, that a similar idea of a median but restricted to particular distances and interval data was discussed in [173, 175], while for fuzzy numbers and L^1-type distance in [174]. On the other hand, a completely different idea of a median in the setting of imprecise probabilities was discussed in [79].

5.2.3 Existence of the fuzzy median

A main result relating to the concepts introduced in the previous section can be obtained using the following well-known embedding theorem of Rådström.

Theorem 5.3. *([166], Theorem 1) Let us consider a triplet* $(\mathbb{X},+,\cdot)$ *where* $+:\mathbb{X}\times\mathbb{X}\to\mathbb{X}$ *and* $\cdot:[0,\infty)\times\mathbb{X}\to\mathbb{X}$ *satisfy the following properties:*

(i) $(a+b)+c=a+(b+c)$ *for all* $a,b,c\in\mathbb{X}$*;*
(ii) $a+b=b+a$ *for all* $a,b\in\mathbb{X}$*;*
(iii) $a+c=b+c$ *implies* $a=b$ *for all* $a,b,c\in\mathbb{X}$*;*
(iv) $\lambda(a+b)=\lambda a+\lambda b$ *for all* $\lambda\in[0,\infty)$ *and* $a,b\in\mathbb{X}$*;*
(v) $(\lambda_1+\lambda_2)a=\lambda_1 a+\lambda_2 a$ *for all* $\lambda_1,\lambda_2\in[0,\infty)$ *and* $a\in\mathbb{X}$*;*
(vi) $\lambda_1(\lambda_2 a)=\lambda_1\lambda_2 a$ *for all* $\lambda_1,\lambda_2\in[0,\infty)$ *and* $a\in\mathbb{X}$*;*
(vii) $1a=a$ *for all* $a\in\mathbb{X}$*.*

Then there exist a vector space $(\widetilde{\mathbb{X}},\oplus,\odot)$ *and an injective application (inclusion)* $i:\mathbb{X}\to\widetilde{\mathbb{X}}$ *and, regarding* $\mathbb{X}$ *as a subset of* $\widetilde{\mathbb{X}}$ *(that is adopting the convention* $i(x)=x$ *for all* $x\in\mathbb{X}$*) we have*

$$a\oplus b=a+b;$$
$$\lambda\odot a=\lambda\cdot a$$

for all $a,b\in\mathbb{X}$ *and* $\lambda\in[0,\infty)$*.*

If, in addition, there exits a metric d defined on $\mathbb{X}$ *satisfying*

(viii) $d(a+c,b+c)=d(a,b)$ *for all* $a,b,c\in\mathbb{X}$*;*
(ix) $d(\lambda a,\lambda b)=\lambda d(a,b)$ *for all* $\lambda\in[0,\infty)$ *and* $a,b\in\mathbb{X}$*;*
(x) $+:\mathbb{X}\times\mathbb{X}\to\mathbb{X}$ *and* $\cdot:[0,\infty)\times\mathbb{X}\to\mathbb{X}$ *are continuous on the topology generated by d on* $\mathbb{X}$*,*

then there exists a norm $\|\cdot\|:\widetilde{\mathbb{X}}\to[0,\infty)$ *such that the metric* $\widetilde{d}$ *on* $\widetilde{\mathbb{X}}$ *generated by* $\|\cdot\|$ *satisfies*

$$d(a,b)=\widetilde{d}(a,b) \text{ for all } a,b\in\mathbb{X}.$$

Spaces like $\mathbb{X}$ satisfying requirements $(i)-(vii)$ in the above theorem are called semilinear spaces.

It is easily seen that the above hypotheses hold for all the metrics proposed in Section 1.5.

Corollary 5.3. *There exists a vector space* $\left(\widetilde{\mathbb{F}(\mathbb{R})}, \oplus, \odot\right)$ *so that*

$$A + B = A \oplus B \text{ for all } A, B \in \mathbb{F}(\mathbb{R})$$

and

$$\lambda A = \lambda \odot A \text{ for all } \lambda \in [0, \infty) \text{ and } A \in \mathbb{F}(\mathbb{R}).$$

Moreover, if D *is a metric on* $\mathbb{F}(\mathbb{R})$*, like those in Section 1.5, then there exists a metric* $\widetilde{D}$ *which makes* $\left(\widetilde{\mathbb{F}(\mathbb{R})}, \oplus, \odot, \widetilde{D}\right)$ *a normed space and such that*

$$D(A,B) = \widetilde{D}(A,B) \text{ for all } A, B \in \mathbb{F}(\mathbb{R}).$$

Next two lemmas concerning the existence of the best approximation will be also useful.

Lemma 5.1. *([176], Theorem 4.1.1) Let* $(\mathbb{X}, d)$ *be a metric space and let* B *be a compact subset of* $\mathbb{X}$*. Then for any* $x \in \mathbb{X}$ *there exists* $x^* \in B$ *such that*

$$d(x, x^*) = \inf_{y \in B} d(x, y).$$

Lemma 5.2. *Let* $(\mathbb{X}, d)$ *be a normed space and let* Ω *be a nonempty closed subset of a linear subspace* $\mathbb{X}_1$ *of* $\mathbb{X}$*. Then, for any* $x \in \mathbb{X}$ *there exists* $x^* \in \Omega$ *such that*

$$d(x, x^*) = \inf_{y \in \Omega} d(x, y). \tag{5.15}$$

The proof can be found in [73], Theorem 3.2.3. Let us continue with several auxiliary results useful for our purposes.

Theorem 5.4. *(see [73], Theorem 3.2.5) Let* d *be a metric defined on the space of fuzzy numbers* $\mathbb{F}(\mathbb{R})$ *which satisfies requirements (viii)-(x) of Theorem 5.3 and let* $\left(\widetilde{\mathbb{F}(\mathbb{R})}, \widetilde{d}, \oplus, \odot\right)$ *be the normed space which realizes the embedding of* $(\mathbb{F}(\mathbb{R}), d, +, \cdot)$ *according to Theorem 5.3. Let us consider a subset* $\mathscr{A} \subseteq \mathbb{F}(\mathbb{R})$ *for which there exist* $\{v_2, v_3, ..., v_m\} \subseteq \mathscr{A}$ *such that:*

(i) *The system* $\{1, v_2, \ldots, v_n\}$ *is linear independent in the vector space* $\left(\widetilde{\mathbb{F}(\mathbb{R})}, \oplus, \odot\right)$.

(ii) $\mathscr{A} = \{\lambda_1 \cdot 1 + \sum_{i=2}^{n} \lambda_i v_i : \lambda_1 \in \mathbb{R}, \lambda_i \in [0, \infty), i = 2, \ldots, n\}$.

Then $\mathscr{A}$ *is a closed subset of* $\mathbb{F}(\mathbb{R})$ *in the topology generated by the metric* d.

Proof. Let $(A_n)_{n \geq 1}$ be a convergent sequence of fuzzy numbers with respect to metric d and such that $A_n \in \mathscr{A}$ for all $n \in \mathbb{N}$, $n \geq 1$. Then let $A_0 = (d) \lim_{n \to \infty} A_n$. We prove that $A_0 \in \mathscr{A}$ and this will end the proof.

We start by rewriting the set $\mathscr{A}$ in a more suitable way. Let us choose arbitrary $A \in \mathscr{A}$ and let $\lambda_1 \in \mathbb{R}$ and $\lambda_i \in [0,\infty)$, $i = 2,\dots,m$, be such that

$$A = \lambda_1 \cdot 1 + \sum_{k=2}^{m} \lambda_i v_i.$$

By Theorem 5.3 it is immediate that $\sum_{k=2}^{m} \lambda_i v_i = \sum_{k=2}^{m} \lambda_i \otimes v_i$. Next, we prove that $\lambda_1 \cdot 1 = \lambda_1 \otimes 1$. We may suppose that $\lambda_1 < 0$ because we already know that in the remaining case the property holds. Let 0 be the null element of $\mathbb{F}(\mathbb{R})$ and let $\widetilde{0}$ be the null element of $\widetilde{\mathbb{F}(\mathbb{R})}$. By elementary reasonings we observe that $0 \oplus 0 = 0 + 0 = 0$ and $0 \oplus \widetilde{0} = 0$. Clearly, we obtain $0 = \widetilde{0}$. This implies that $1 \oplus (-1) = 1 + (-1) = 0 = \widetilde{0}$, i.e. -1 is the opposite to 1 with respect to $\oplus$ too. Hence

$$\lambda_1 \cdot 1 = (-\lambda_1) \cdot (-1) = (-\lambda_1) \otimes (-1) = \lambda_1 \otimes 1.$$

Consequently, we obtain

$$\mathscr{A} = \left\{ \lambda_1 \otimes 1 + \sum_{k=2}^{m} \lambda_i \otimes v_i : \lambda_1 \in \mathbb{R} \text{ and } \lambda_i \in [0,\infty), i = 2,\dots,m \right\}. \tag{5.16}$$

This implies that $\mathscr{A} \subseteq \mathscr{L}$ where $\mathscr{L} = span\{1, v_2, \dots, v_m\}$. Now, since $A_0 = (d) \lim\limits_{n\to\infty} A_n$ implies that $A_0 = (\widetilde{d}) \lim\limits_{n\to\infty} A_n$ where $A_n \in \mathscr{L}$, $n \geq 1$, and since it is well known that any finite dimensional linear subspace of a vector space is a closed set with respect to any norm generated topology, it easily follows that $A_0 \in \mathscr{L}$. Therefore, let $\lambda_i(A_0) \in \mathbb{R}$, $i = 1,\dots,m$, be such that

$$A_0 = \lambda_1(A_0) \otimes 1 + \sum_{k=2}^{m} \lambda_i(A_0) \otimes v_i.$$

For each $n \in \mathbb{N}$, $n \geq 1$, let $\lambda_i(A_n)$, $i = 1,\dots,m$, be such that

$$A_n = \lambda_1(A_n) \otimes 1 + \sum_{k=2}^{m} \lambda_i(A_n) \otimes v_i.$$

Further on it will serve to define an Euclidean type metric on $\mathscr{L}$. Let $A, B \in \mathscr{L}$ be arbitrarily chosen and let $\lambda_i(A), \lambda_i(B)$, $i = 1,\dots,m$, be such that

$$A = \lambda_1(A) \otimes 1 + \sum_{k=2}^{m} \lambda_i(A) \otimes v_i,$$
$$B = \lambda_1(B) \otimes 1 + \sum_{k=2}^{m} \lambda_i(B) \otimes v_i.$$

We set

$$D^2(A,B) = \sum_{k=1}^{m} (\lambda_i(A) - \lambda_i(B))^2.$$

It is immediate that D defines a metric on $\mathscr{L}$. One can easily prove that $(\mathscr{L}, D)$ is a normed space. It is well-known in topology that all the norm generated topological spaces defined on a finite dimensional vector space are equivalent. Therefore, $\widetilde{d}$ and D generate the same topology on $\mathscr{L}$. This means that since $(\widetilde{d}) \lim_{n\to\infty} A_n = A_0$, we also have $(D) \lim_{n\to\infty} A_n = A_0$. This implies

$$\lim_{n\to\infty} \sum_{k=1}^{m} (\lambda_i(A_n) - \lambda_i(A_0))^2 = 0$$

and then we easily obtain that $\lim_{n\to\infty} \lambda_i(A_n) = \lambda_i(A_0)$ for all $i = 1, \ldots, m$. On the other hand, since $\{A_n : n \in \mathbb{N}, n \geq 1\} \subseteq \mathscr{A}$, we have $\lambda_i(A_n) \geq 0$ for all $n \in \mathbb{N}$, $n \geq 1$ and for all $i = 2, 3, \ldots, m$. Consequently $\lim_{n\to\infty} \lambda_i(A_n) \geq 0$ for all $i = 2, 3, \ldots, m$, i.e. $\lambda_i(A_0) \geq 0$ for all $i = 2, 3, \ldots, m$. Summing up, we have

$$A_0 = \lambda_1(A_0) \otimes 1 + \sum_{k=2}^{m} \lambda_i(A_0) \otimes v_i,$$

where $\lambda_1(A_0) \in \mathbb{R}$ and $\lambda_i(A_0) \in [0, \infty)$ for all $i = 2, 3, \ldots, m$. Thus we can write

$$A_0 = \lambda_1(A_0) \cdot 1 + \sum_{k=2}^{m} \lambda_i(A_0) \cdot v_i.$$

Hence it easily follows that $A_0 \in \mathscr{A}$ and the theorem is proved. ■

Corollary 5.4. *([73], Corollary 3.2.6) Let d be a metric defined on the space of fuzzy numbers $\mathbb{F}(\mathbb{R})$ which satisfies requirements $(viii) - (x)$ of Theorem 5.3. Let us consider a subset $\mathscr{A} \subseteq \mathbb{F}(\mathbb{R})$ which satisfies requirements $(i) - (ii)$ of Theorem 5.4. If $\left(\widetilde{\mathbb{F}(\mathbb{R})}, \widetilde{d}\right)$ is a normed space, where $\widetilde{d}$ is the extension of the metric d on $\widetilde{\mathbb{F}(\mathbb{R})}$ according to Theorem 5.3, then $\mathscr{A}$ is a closed subset of $\widetilde{\mathbb{F}(\mathbb{R})}$ in the topology generated by $\widetilde{d}$ on $\widetilde{\mathbb{F}(\mathbb{R})}$.*

Proof. The desired conclusion easily follows from the last lines of the proof of Theorem 5.4 which are just below relation (5.16). Indeed, the only difference in the reasoning is that now A_0 is supposed to be in $\widetilde{\mathbb{F}(\mathbb{R})}$. But since $\mathscr{L}$ is closed in $\widetilde{\mathbb{F}(\mathbb{R})}$ with respect to the topology generated by $\widetilde{d}$, we conclude that $A_0 \in \mathscr{L}$ and hence $A_0 \in \mathscr{A}$. ■

Note that in general if $(\mathbb{X}, \delta)$ is a topological space, $\mathbb{X}_1 \subseteq \mathbb{X}$ and if A is a closed subset of $\mathbb{X}_1$ in the topology induced by δ on $\mathbb{X}_1$, then A is not necessarily a closed subset of X in the topology δ. Therefore, the form of the set $\mathscr{A}$ in Theorem 5.4 is important in order to obtain the conclusion of the above corollary.

Now, combining Lemma 5.2 and Corollary 5.4, we easily obtain the following corollary.

Corollary 5.5. *([73], Corollary 3.2.8) Let d be a metric defined on $\mathbb{F}(\mathbb{R})$ which satisfies requirements $(viii)-(x)$ of Theorem 5.3 and let $\mathscr{A}$ be a subset of $\mathbb{F}(\mathbb{R})$ like those from the assumptions of Theorem 5.4. Then for any $A \in \mathbb{F}(\mathbb{R})$ there exists $A^* \in \mathscr{A}$ such that*

$$d(A,A^*) = \inf_{B\in\mathscr{A}} d(A,B).$$

Proof. By Corollary 5.4 $\left(\widetilde{\mathbb{F}(\mathbb{R})}, \widetilde{d}\right)$ is a normed space, where $\widetilde{d}$ is the extension of the metric d on $\widetilde{\mathbb{F}(\mathbb{R})}$ according to Theorem 5.3. y Corollary 5.4 it also results that $\mathscr{A}$ is a closed subset of $\mathscr{L}$ (in the topology generated by $\widetilde{d}$ on $\widetilde{\mathbb{F}(\mathbb{R})}$), where $\mathscr{L}$ denotes the same finite dimensional linear subspace of $\widetilde{\mathbb{F}(\mathbb{R})}$, like in the proof of Theorem 5.4. By Theorem 5.2 it is immediate that for any $A \in \widetilde{\mathbb{F}(\mathbb{R})}$ there exists $A^* \in \mathscr{A}$ such that $\widetilde{d}(A,A^*) = \inf_{B\in\mathscr{A}} \widetilde{d}(A,B)$. In particular, if $A \in \mathbb{F}(\mathbb{R})$ then noting that $\widetilde{d}(A,B) = d(A,B)$ for all $B \in \mathscr{A}$ we easily obtain the desired conclusion. ■

Keeping in mind the fuzzy medians defined in Definitions 5.1-5.4 we apply Corollary 5.5 for some classes of fuzzy numbers most interesting for us.

Theorem 5.5. *([73], Theorem 3.2.9) Let Ω be one of the following subsets of $\mathbb{F}(\mathbb{R})$, $\mathbb{F}^T(\mathbb{R})$, $\mathbb{F}^\Delta(\mathbb{R})$, $\mathbb{I}$ or $\mathbb{R}$. If d is a metric on $\mathbb{F}(\mathbb{R})$ which satisfies requirements $(viii)-(x)$ of Theorem 5.3 then Ω is a closed subset of $\mathbb{F}(\mathbb{R})$ in the topology generated by d. In addition, if $\left(\widetilde{\mathbb{F}(\mathbb{R})}, \widetilde{d}\right)$ is a normed space, where $\widetilde{d}$ is the extension of the metric d on $\widetilde{\mathbb{F}(\mathbb{R})}$ according to Theorem 5.3, then Ω is a closed subset of $\widetilde{\mathbb{F}(\mathbb{R})}$ in the topology generated by $\widetilde{d}$ on $\widetilde{\mathbb{F}(\mathbb{R})}$.*

Proof. The proof of the first statement is immediate since Ω has the properties of the set $\mathscr{A}$ described in Theorem 5.4. Suppose, e.g. that $\Omega = \mathbb{F}^T$. Since any $T = (t_1,t_2,t_3,t_4) \in \mathbb{F}^T(\mathbb{R})$ can be written as

$$T = t_1 \cdot 1 + (t_2 - t_1)\, v_1 + (t_3 - t_2)\, v_3 + (t_4 - t_3)\, v_4,$$

where $v_2 = (0,1,1,1)$, $v_3 = (0,0,1,1)$, $v_4 = (0,0,0,1)$, we can take $\mathscr{A} = \mathbb{F}^T(\mathbb{R})$ in Theorem 5.4. Similarly, we get the same conclusion for the other cases.

The proof of the second statement is immediate since by the similar reasoning we can show that Ω satisfies assumptions of Corollary 5.4. ■

The following result will be very useful in proving the existence of the fixed-shape fuzzy median.

Theorem 5.6. *([73], Theorem 3.2.12) Let Ω be one of the following subsets of $\mathbb{F}(\mathbb{R})$, $\mathbb{F}^T(\mathbb{R})$, $\mathbb{F}^\Delta(\mathbb{R})$, $\mathbb{I}$ or $\mathbb{R}$. If d is a metric on $\mathbb{F}(\mathbb{R})$ satisfying requirements $(viii)-(x)$ of Theorem 5.3 then for any $A \in \Omega$ there exits $A^* \in \Omega$ such that*

$$d(A,A^*) = \inf_{B\in\Omega} d(A,B).$$

Proof. By the proof of Theorem 5.5 one can see that Ω can be represented as a set $\mathscr{A}$ considered in Theorem 5.4. Therefore, the desired conclusion easily follows from Corollary 5.5. ■

Just before the main theorem we still need some further constructions. Let us consider a metric d defined on the space $\mathbb{F}(\mathbb{R})$ which satisfies requirements $(viii)-(x)$ of Theorem 5.3 and let $\left(\widetilde{\mathbb{F}(\mathbb{R})}, \widetilde{d}, \oplus, \odot\right)$ be the normed space which realizes the embedding of $(\mathbb{F}(\mathbb{R}), d, +, \cdot)$. We construct now the power spaces of $\mathbb{F}(\mathbb{R})$ and $\widetilde{\mathbb{F}(\mathbb{R})}$ denoted by $\mathbb{F}(\mathbb{R})^n$ and $\widetilde{\mathbb{F}(\mathbb{R})}^n$, respectively, where $n \in \mathbb{N}$, $n \geq 2$ is fixed. By $A \in \mathbb{F}(\mathbb{R})^n$ we agree that $A = (A_1, \ldots, A_m)$, where $A_i \in \mathbb{F}(\mathbb{R})$ for all $i = 1, \ldots, n$. We adopt the same convention in the case of $\widetilde{\mathbb{F}(\mathbb{R})}$. Further on, we construct on $\widetilde{\mathbb{F}(\mathbb{R})}^n$ a vector space structure by extending the vector space structure of $\widetilde{\mathbb{F}(\mathbb{R})}$ using the well-known standard procedure. Then we notice that since $\mathbb{F}(\mathbb{R}) \subseteq \widetilde{\mathbb{F}(\mathbb{R})}$, we also have $\mathbb{F}(\mathbb{R})^n \subseteq \widetilde{\mathbb{F}(\mathbb{R})}^n$. We define metrics δ on $\mathbb{F}(\mathbb{R})^n$ and $\widetilde{\delta}$ on $\widetilde{\mathbb{F}(\mathbb{R})}^n$, where

$$\delta(A,B) = \sum_{i=1}^{n} d(A_i, B_i)$$

and

$$\widetilde{\delta}(A,B) = \sum_{i=1}^{n} \widetilde{d}(A_i, B_i).$$

Then for any $A, B \in \mathbb{F}(\mathbb{R})^n$ we have $\widetilde{\delta}(A,B) = \delta(A,B)$. We are interested in a particular kind of set which will help us later to prove the existence of the fuzzy median. Suppose that $\mathscr{A}$ is a subset of $\mathbb{F}(\mathbb{R})$ like those assumed in Theorem 5.4. We introduce the diagonal set of $\mathscr{A}$ in $\mathbb{F}(\mathbb{R})^n$ given by

$$D^n(\mathscr{A}) = \{(A, \ldots, A) : A \in \mathscr{A}\}.$$

Since $\mathscr{A}$ is closed in $\mathbb{F}(\mathbb{R})$ it easily follows that $D^n(\mathscr{A})$ is closed in $\mathbb{F}(\mathbb{R})^n$. Indeed, to prove this fact let us consider a sequence $(\overline{A}_k)_{k \geq 1}$ in $D^n(\mathscr{A})$ such that $(\delta) \lim_{k \to \infty} \overline{A}_k = \overline{A}_0$. Let us adopt the following notation:

$$\overline{A}_k = (A_k, \ldots, A_k), \quad k \in \mathbb{N}, \quad k \geq 1$$

and

$$\overline{A}_0 = (A_0^1, A_0^2, \ldots, A_0^n).$$

By the convergence property and the definition of the metric δ we get $(d) \lim_{k \to \infty} \overline{A}_n = A_0^i$, $i = 1, \ldots, n$. Then, since $\mathscr{A}$ is closed in $\mathbb{F}(\mathbb{R})$, we conclude that $A_0^i \in \mathscr{A}$ for all $i = 1, \ldots, n$. The uniqueness of the limit implies $A_0^1 = \ldots = A_0^n$ and thus we obtain $\overline{A}_0 \in D^n(\mathscr{A})$. Clearly, this implies that $D^n(\mathscr{A})$ is closed in $\mathbb{F}(\mathbb{R})^n$. Analogously, if $\mathscr{A}$ is a closed subset of $\widetilde{\mathbb{F}(\mathbb{R})}$ then $D^n(\mathscr{A})$ is a closed subset of $\widetilde{\mathbb{F}(\mathbb{R})}^n$.

We are now in position to give the sufficient conditions for the existence of the fuzzy median.

Theorem 5.7. *([38], Theorem 14) Let d be a metric defined on $\mathbb{F}(\mathbb{R})$ which satisfies requirements $(viii)-(x)$ of Theorem 5.3. Let us consider an arbitrary sample of fuzzy numbers $A_1,\dots,A_n$ and let Ω be one of the following subsets of $\mathbb{F}(\mathbb{R})$, $\mathbb{F}^T(\mathbb{R})$, $\mathbb{F}^\Delta(\mathbb{R})$, $\mathbb{I}$ or $\mathbb{R}$. Then there exits a fuzzy median of the sample with respect to Ω and the metric d.*

Proof. By the proof of Theorem 5.5 it results that Ω satisfies assumptions of Theorem 5.4 and Corollary 5.4 by taking $\mathscr{A}=\Omega$. Analyzing the form of $\mathscr{A}$ (and hence Ω) in Theorem 5.4 it is easily seen that Ω is a closed subset of a finite dimensional linear subspace $\mathscr{L}\subseteq\widetilde{\mathbb{F}(\mathbb{R})}$. Now, since d satisfies requirements $(viii)-(x)$ of Theorem 5.3, let $\left(\widetilde{\mathbb{F}(\mathbb{R})},\widetilde{d},\oplus,\odot\right)$ be a normed space which realizes the embedding of $(\mathbb{F}(\mathbb{R}),d,+,\cdot)$. Thus the power space $\left(\widetilde{\mathbb{F}(\mathbb{R})}^n,\widetilde{\delta},\oplus,\odot\right)$ is a normed space. Taking into account the properties of Ω we conclude that $D^n(\Omega)$ is a closed convex subset of the linear subspace $\mathscr{L}^n\subseteq\widetilde{\mathbb{F}(\mathbb{R})}^n$. For $A=(A_1,\dots,A_n)$ Lemma 5.2 guarantees the existence of $\overline{A}_0\in D^n(\Omega)$, $\overline{A}_0=(A_0,\dots,A_0)$, such that

$$\widetilde{\delta}(A,\overline{A}_0)\le\widetilde{\delta}(A,\overline{B}) \tag{5.17}$$

for all $\overline{B}\in D^n(\Omega)$. Now, let us choose arbitrary $B\in\Omega$. We observe that $\overline{B}\in D^n(\Omega)$, where $\overline{B}=(B,\dots,B)$. Taking into account the properties of $\widetilde{\delta}$ and $\widetilde{d}$ we get

$$\sum_{i=0}^{n}d(A_i,A_0)=\widetilde{\delta}(A,\overline{A}_0)$$

and

$$\sum_{i=0}^{n}d(A_i,B)=\widetilde{\delta}(A,\overline{B}),$$

which by relation (5.17) implies

$$\sum_{i=0}^{n}d(A_i,A_0)\le\sum_{i=0}^{n}d(A_i,B).$$

This means that A_0 is a fuzzy median of the sample $A_1,\dots,A_n$ with respect to Ω and metric d. ■

Note that in the above theorem the convexity of Ω is not needed for the existence result because in Lemma 5.2 the convexity is not required. It is worth stressing that all the distances or families of distances between fuzzy numbers discussed in Section 1.5 satisfy the assumptions of Theorem 5.7. Hence the fixed-shape fuzzy median always exists with respect to those distances. Finally, we have to notice that even if we proved the existence of the fuzzy median, in applications one requires its location or at least its good approximation. We have an ongoing research in this respect and the results are very promising.

5.3 Fuzzy numbers in FMDM

5.3.1 Introduction to FMDM

A standard multicriteria decision making (MDM) problem assumes the evaluation and ranking of m alternatives $A_1,\ldots,A_m$ under n criteria $C_1,\ldots,C_n$ by a committee of k decision makers $D_1,\ldots,D_k$. We assume $C_1,\ldots,C_h$ denote subjective criteria, $C_{h+1},\ldots,C_p$ objective criteria like benefit (i.e. the larger the better) and $C_{p+1},\ldots,C_n$ are objective criteria like costs (i.e. the smaller the better). We denote by r_{ijt}, where $i=1,\ldots,m,\ j=1,\ldots,h$ and $t=1,\ldots,k$, the performance of alternative A_i versus subjective criterion C_j in the opinion of the decision maker D_t and by x_{ij}, where $i=1,\ldots,m,\ j=h+1,\ldots,n$, the performance of alternative A_i versus objective criterion C_j. The weights of the criteria are often imposed by the decision makers. By w_{jt}, where $j=1,\ldots,n$ and $t=1,\ldots,k$, we denote the weight of the criterion C_j in the opinion of the decision maker D_t.

Standard MDM methods cannot often be applied due to uncertain or incomplete information. Fuzzy numbers have been already accepted as an useful tool for modeling such information. Therefore, if for some reasons the evaluations r_{ijt}, performances x_{ij} and/or weights w_{jt} are expressed by fuzzy sets, particularly by fuzzy numbers , then we have a *fuzzy multicriteria decision making* (FMDM) problem which needs a specific attitude and solution. Hundreds of methods related to FMDM were proposed in the literature, e.g., [8, 23, 60, 68, 92, 133, 140, 197, 200].

Let us start by considering the following example (see [68, 23]).

Example 5.2. A company must select a distribution center from the following three centers A_1,A_2,A_3 to serve better its customers. Four decision-makers D_1,D_2,D_3,D_4, four subjective criteria (C_1 - transportation availability, C_2 - human resource, C_3 - market potential and C_4 - climate condition) and one objective criterion (C_5 - cost in million US$) are considered. The decision-makers use fuzzy terms (modeled by triangular fuzzy numbers) from the following linguistic rating set $S=\{VP,P,F,G,VG\}$, where

$$\begin{aligned}
VP &= \text{Very Poor} = (0,0,0.2)\,,\\
P &= \text{Poor} = (0,0.2,0.4)\,,\\
F &= \text{Fair} = (0.3,0.5,0.7)\,,\\
G &= \mathit{Good} = (0.6,0.8,1)\,,\\
VG &= \text{Very Good} = (0.8,1,1)
\end{aligned}$$

to evaluate the subjective criteria C_1, C_2, C_3, C_4 and a linguistic weighting set $W = \{VL, L, M, H, VH\}$, where

$$\begin{aligned} VL &= \text{Very Low} = (0,0,0.3)\,, \\ L &= \text{Low} = (0,0.3,0.5)\,, \\ M &= \text{Medium} = (0.2,0.5,0.8)\,, \\ H &= \text{High} = (0.5,0.7,1)\,, \\ VH &= \text{Very High} = (0.7,1,1) \end{aligned}$$

to assess the importance of criteria C_1, C_2, C_3, C_4, C_5. The ratings of alternatives versus criteria under the opinion of decision-makers are presented in Table 5.1 and the importance weights of the five criteria from the four decision-makers are displayed in Table 5.2. □

Table 5.1 Ratings of alternatives versus criteria (see Example 5.2).

Criteria/	Decision-makers				r_{ij}
Alternatives	D_1	D_2	D_3	D_4	
C_1/A_1	G	G	VG	G	$(0.650, 0.850, 1.000)$
C_1/A_2	G	VG	F	F	$(0.500, 0.700, 0.850)$
C_1/A_3	VG	G	G	G	$(0.650, 0.850, 1.000)$
C_2/A_1	G	F	VG	G	$(0.575, 0.775, 0.925)$
C_2/A_2	F	G	VG	VG	$(0.625, 0.825, 0.925)$
C_2/A_3	F	F	G	F	$(0.375, 0.575, 0.775)$
C_3/A_1	VG	G	G	G	$(0.650, 0.850, 1.000)$
C_3/A_2	G	F	VG	G	$(0.575, 0.775, 0.925)$
C_3/A_3	F	F	G	G	$(0.450, 0.650, 0.850)$
C_4/A_1	F	P	F	F	$(0.225, 0.425, 0.625)$
C_4/A_2	F	F	G	G	$(0.450, 0.650, 0.850)$
C_4/A_3	G	F	G	F	$(0.450, 0.650, 0.850)$
C_5/A_1	$(6.0, 7.0, 7.5)$				$(0.480, 0.514, 0.600)$
C_5/A_2	$(3.6, 4.0, 4.8)$				$(0.750, 0.900, 1.000)$
C_5/A_3	$(4.7, 5.0, 5.6)$				$(0.643, 0.720, 0.766)$

Table 5.2 The importance weights of the criteria and the aggregated weights (see Example 5.2).

Criteria	Decision-makers				w_j
	D_1	D_2	D_3	D_4	
C_1	VH	VH	H	VH	$(0.650, 0.925, 1.000)$
C_2	L	M	M	M	$(0.150, 0.450, 0.725)$
C_3	L	L	M	M	$(0.100, 0.400, 0.650)$
C_4	M	H	VH	VH	$(0.525, 0.800, 0.950)$
C_5	H	VH	VH	H	$(0.600, 0.850, 1.000)$

The expected value (1.72) has many interesting properties that make it useful in many areas, like aggregation and decision support. It is also a very good ranking index, as it was proved in [36]. The method of ranking fuzzy numbers used in the present section was introduced in [12] (see also Section 4.2) as follows

$$A \succ B \text{ if and only if } EV(A) > EV(B), \tag{5.18}$$

$$A \prec B \text{ if and only if } EV(A) < EV(B), \tag{5.19}$$

$$A \sim B \text{ if and only if } EV(A) = EV(B) \tag{5.20}$$

$$A \succeq B \text{ if and only if } EV(A) \geq EV(B). \tag{5.21}$$

Because of its properties and easiness in calculations the expected value seems to have suitable properties to be applied in fuzzy multicriteria decision making.

5.3.2 Algorithms useful in FMDM

FMDM methods known from the literature are often elaborated for the particular case of trapezoidal or triangular fuzzy numbers. Nevertheless, as it was pointed out in [23], the general case is completely justified if we have more exact information about the performance of alternatives versus criteria and/or we have some difficulties to select the best alternative (alternatives) because the ratings are too close. For example, it may be suitable to replace the trapezoidal fuzzy number (a,b,c,d) which describes the criteria of investment cost with the semi-trapezoidal fuzzy number $(a,b,c,d)_{r,s}$ with $r,s > 1$ (see (1.15), (1.16)), if the cost lies almost sure between b and c (more we are sure that the investment cost is between b and c, both r and s may be higher) or with $(a,b,c,d)_{r,s}$ with $0 < r,s < 1$, if there is a relatively great possibility that the cost would be somewhere in $[a,b] \cup [c,d]$ (if there are some doubts that the investment cost is between b and c both r and s may be lower). The basic rating of an alternative with respect to a subjective criterion given by the trapezoidal fuzzy number (a,b,c,d) can be refined too in the same way. The rating $(a,b,c,d)_{r,s}$ with $r,s > 1$ means that the decision-maker is almost sure that the crisp rating is in $[b,c]$ but he gives a little possibility as the rating is in $[a,b] \cup [c,d]$. The rating $(a,b,c,d)_{r,s}$ with $r,s < 1$ means that in the opinion of the decision-maker there is a relatively great possibility that the crisp rating is out of $[b,c]$, in $[a,b]$ or $[c,d]$. The rating $(a,b,c,d)_{r,s}$ with $r > 1, s < 1$ means that the possibility as the rating is in $[a,b]$ is low and the possibility as the rating is in $[c,d]$ is relatively great. The rating $(a,b,c,d)_{r,s}$ with $r < 1, s > 1$ means that the possibility as the rating is in $[a,b]$ is great and the possibility as the rating is in $[c,d]$ is relatively low.

Having in mind the notation introduced above, a fuzzy number w_{jt}, where $j = 1,\ldots,n$ and $t = 1,\ldots,k$, given by α-cuts

$$(w_{jt})_\alpha = \left[(w_{jt})_L(\alpha), (w_{jt})_U(\alpha)\right], \quad \alpha \in [0,1],$$

denotes the importance weight of the criterion C_j in the opinion of the decision-maker D_t. A fuzzy number r_{ijt}, where $i = 1,\ldots,m,\ j = 1,\ldots,h,\ t = 1,\ldots,k$, such that

$$(r_{ijt})_\alpha = \left[(r_{ijt})_L(\alpha), (r_{ijt})_U(\alpha)\right], \quad \alpha \in [0,1],$$

denotes the performance of the alternative A_i versus subjective criterion C_j in the opinion of the decision-maker D_t. A fuzzy number x_{ij}, where $i = 1,\ldots,m,\ j = h+1,\ldots,n$, given by

$$(x_{ij})_\alpha = \left[(x_{ij})_L(\alpha), (x_{ij})_U(\alpha)\right], \quad \alpha \in [0,1],$$

describes the performance of the alternative A_i versus objective criterion C_j.

We suggest to compute the normalized value r_{ij} of a fuzzy number x_{ij} by

$$r_{ij} = \frac{1}{\Delta_j} \cdot \left(x_{ij} \ominus \min_{i=1,\ldots,m} (x_{ij})_L(0) \right), \tag{5.22}$$

if $j = h+1,\ldots,p$, i.e. C_j is a objective criterion of benefit kind or

$$r_{ij} = \frac{1}{\Delta_j} \cdot \left(\max_{i=1,\ldots,m} (x_{ij})_U(0) \ominus x_{ij} \right), \tag{5.23}$$

if $j = p+1,\ldots,n$, i.e. C_j is a objective criterion of cost kind, where

$$\Delta_j = \max_{i=1,\ldots,m} (x_{ij})_U(0) - \min_{i=1,\ldots,m} (x_{ij})_L(0), j = h+1,\ldots,n$$

and (see (1.24))

$$(A \ominus z)_\alpha = [A_L(\alpha) - z, A_U(\alpha) - z],$$
$$(z \ominus A)_\alpha = [z - A_U(\alpha), z - A_L(\alpha)],$$

for every $A \in \mathbb{F}(\mathbb{R})$, $z \in \mathbb{R}$ and $\alpha \in [0,1]$.

The averaged weight of C_j assessed by decision-makers $D_1,\ldots,D_k$ is given by

$$w_j = \frac{1}{k} \cdot \left(w_{j1} + \ldots + w_{jk}\right), \quad j = 1,\ldots,n,$$

i.e. (see (1.21) and (1.29))

$$(w_j)_L(\alpha) = \frac{1}{k}\left(\left(w_{j1}\right)_L(\alpha) + \ldots + \left(w_{jk}\right)_L(\alpha)\right),$$
$$(w_j)_U(\alpha) = \frac{1}{k}\left(\left(w_{j1}\right)_U(\alpha) + \ldots + \left(w_{jk}\right)_U(\alpha)\right).$$

The final fuzzy number evaluation G_i of the alternative A_i is the aggregation of the weighted ratings given by

$$G_i = \frac{1}{n} \cdot \left((r_{i1} \cdot w_1) + \ldots + (r_{in} \cdot w_n) \right), \quad i = 1, \ldots, m.$$

Taking into account the above considerations and (5.18)-(5.20) the following procedure can be used to rank the alternatives $A_1, \ldots, A_m$.

Algorithm 4 *(see [23])*

Step 1. *Compute* $(r_{ij})_\alpha = \left[(r_{ij})_L(\alpha), (r_{ij})_U(\alpha) \right]$ *for* $i = 1, \ldots, m$ *and* $j = 1, \ldots, h$; $\alpha \in [0,1]$, *where*

$$(r_{ij})_L(\alpha) = \frac{1}{k} \sum_{t=1}^{k} (r_{ijt})_L(\alpha)$$

$$(r_{ij})_U(\alpha) = \frac{1}{k} \sum_{t=1}^{k} (r_{ijt})_U(\alpha).$$

Step 2. *Compute* $(r_{ij})_\alpha = \left[(r_{ij})_L(\alpha), (r_{ij})_U(\alpha) \right]$ *for* $i = 1, \ldots, m$ *and* $j = h+1, \ldots, n$; $\alpha \in [0,1]$, *where*

$$(r_{ij})_L(\alpha) = \frac{(x_{ij})_L(\alpha) - \min\limits_{i=1,\ldots,m} (x_{ij})_L(0)}{\Delta_j},$$

$$(r_{ij})_U(\alpha) = \frac{(x_{ij})_U(\alpha) - \min\limits_{i=1,\ldots,m} (x_{ij})_L(0)}{\Delta_j}$$

if $j = h+1, \ldots, p$, *or*

$$(r_{ij})_L(\alpha) = \frac{\max\limits_{i=1,\ldots,m} (x_{ij})_U(0) - (x_{ij})_U(\alpha)}{\Delta_j},$$

$$(r_{ij})_U(\alpha) = \frac{\max\limits_{i=1,\ldots,m} (x_{ij})_U(0) - (x_{ij})_L(\alpha)}{\Delta_j}$$

if $j = p+1, \ldots, n$, *where for any* $j = h+1, \ldots, n$

$$\Delta_j = \max_{i=1,\ldots,m} (x_{ij})_U(0) - \min_{i=1,\ldots,m} (x_{ij})_L(0).$$

Step 3. *Compute* $(w_j)_\alpha = \left[(w_j)_L(\alpha), (w_j)_U(\alpha) \right]$ *for* $j = 1, \ldots, n$; $\alpha \in [0,1]$, *where*

$$(w_j)_L(\alpha) = \frac{1}{k} \sum_{t=1}^{k} (w_{jt})_L(\alpha),$$

$$(w_j)_U(\alpha) = \frac{1}{k} \sum_{t=1}^{k} (w_{jt})_U(\alpha).$$

Step 4. *Compute for every* $j = 1, \ldots, n$

$$EV\left(r_{ij} \cdot w_j\right) = \frac{1}{2}\left(\int_0^1 \left(r_{ij}\right)_L(\alpha)\left(w_j\right)_L(\alpha)\,d\alpha + \int_0^1 \left(r_{ij}\right)_U(\alpha)\left(w_j\right)_U(\alpha)\,d\alpha\right).$$

Step 5. *Calculate for every* $i = 1, \ldots, m$

$$EV\left(G_i\right) = \frac{1}{n}\sum_{j=1}^{n} EV\left(r_{ij} \cdot w_j\right).$$

Step 6. *If* $EV\left(G_{i_1}\right) > EV\left(G_{i_2}\right) > \ldots > EV\left(G_{i_m}\right)$ *then alternatives* $A_{i_1}, A_{i_2}, \ldots, A_{i_m}$ *form a descending order, i.e.* A_{i_1} *is better than* A_{i_2} *and so on till* A_{i_m} *which is the worst alternative.*

Because the effective calculus are complicate enough we prefer a simplified case of the problem considered in Example 5.2 to illustrate Algorithm 4.

Example 5.3. (see [23]) Keeping in mind the notations introduced in Example 5.2 we consider only three criteria C_1, C_2, C_5 and two decision-makers D_1 and D_2. The ratings of alternatives versus criteria and the importance weights are given in Table 5.3 and Table 5.4, respectively. To obtain the fuzzy numbers r_{ij} and w_j for $i = 1, 2, 3$ and $j = 1, 2, 5$ Step 1-Step 3 in Algorithm 4 are applied. It is important to remember here that (see Lemma 1.4)

$$\begin{aligned}(a_1, b_1, c_1, d_1)_{2,2} + (a_2, b_2, c_2, d_2)_{2,2} &= (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)_{2,2}\\ \lambda \cdot (a, b, c, d)_{2,2} &= (\lambda a, \lambda b, \lambda c, \lambda d)_{2,2},\end{aligned}$$

for any semi-trapezoidal fuzzy numbers $(a_1, b_1, c_1, d_1)_{2,2}$ and $(a_2, b_2, c_2, d_2)_{2,2}$ of type $(2,2)$ and $\lambda > 0$. The results are also given in Table 5.3 and Table 5.4. To complete Step 4 we need the following property (see [23], Proposition 1)

$$\begin{aligned}EV\left((a_1, b_1, c_1, d_1)_{2,2} \cdot (a_2, b_2, c_2, d_2)_{2,2}\right) = &\frac{1}{12}\left(a_1 a_2 + 3 b_1 b_2 + 3 c_1 c_2 + d_1 d_2\right)\\ &+ \frac{1}{12}\left(a_1 b_2 + a_2 b_1 + c_1 d_2 + c_2 d_1\right)\end{aligned}$$

valid for every semi-trapezoidal fuzzy numbers $(a_1, b_1, c_1, d_1)_{2,2}$ and $(a_2, b_2, c_2, d_2)_{2,2}$.

Finally we obtain

$$EV\left(G_2\right) = 0.57799 > EV\left(G_3\right) = 0.49826 > EV\left(G_1\right) = 0.36816,$$

which means that A_2 is the best alternative while A_1 is the worst one. □

Table 5.3 Ratings of alternatives versus criteria (see Example 5.3).

Criteria /Alternatives	Decision-makers D_1	D_2	r_{ij}
C_1/A_1	$(0.3,0.5,0.8,1)_{2,2}$	$(0.6,0.8,0.8,1)_{2,2}$	$(0.450,0.650,0.800,1.000)_{2,2}$
C_1/A_2	$(0.6,0.8,0.8,1)_{2,2}$	$(0.8,1,1,1)_{2,2}$	$(0.700,0.900,0.900,1.000)_{2,2}$
C_1/A_3	$(0.8,1,1,1)_{2,2}$	$(0.6,0.8,0.8,1)_{2,2}$	$(0.700,0.900,0.900,1.000)_{2,2}$
C_2/A_1	$(0.6,0.8,0.8,1)_{2,2}$	$(0.3,0.5,0.5,0.7)_{2,2}$	$(0.450,0.650,0.650,0.850)_{2,2}$
C_2/A_2	$(0.3,0.5,0.5,0.7)_{2,2}$	$(0.6,0.8,0.8,1)_{2,2}$	$(0.450,0.650,0.650,0.850)_{2,2}$
C_2/A_3	$(0.3,0.5,0.5,0.7)_{2,2}$	$(0.3,0.5,0.5,0.7)_{2,2}$	$(0.300,0.500,0.500,0.700)_{2,2}$
C_5/A_1	$(6.0,6.5,7.0,7.6)_{2,2}$		$(0,0.150,0.275,0.400)_{2,2}$
C_5/A_2	$(3.6,4.0,4.8,5.0)_{2,2}$		$(0.650,0.700,0.900,1.000)_{2,2}$
C_5/A_3	$(4.7,5.0,5.5,5.6)_{2,2}$		$(0.500,0.525,0.650,0.725)_{2,2}$

Table 5.4 The importance weights of the criteria and the aggregated weights (see Example 5.3).

Criteria	Decision-makers D_1	D_2	w_j
C_1	$(0.7,1,1,1)_{2,2}$	$(0.5,0.7,1,1)_{2,2}$	$(0.600,0.850,1.000,1.000)_{2,2}$
C_2	$(0,0.3,0.3,0.5)_{2,2}$	$(0.2,0.5,0.5,0.8)_{2,2}$	$(0.100,0.400,0.400,0.650)_{2,2}$
C_5	$(0.5,0.7,0.7,1)_{2,2}$	$(0.7,1,1,1)_{2,2}$	$(0.600,0.850,0.850,1.000)_{2,2}$

In the sequel we discuss two particular cases, when the importance weights of criteria and the ratings of alternatives versus all criteria are real numbers.

Taking into account the properties of linearity of the expected interval, i.e.

$$EV(A+B) = EV(A) + EV(B)$$

and

$$EV(\lambda \cdot A) = \lambda EV(A),$$

for every $A, B \in \mathbb{F}(\mathbb{R})$ and $\lambda \in \mathbb{R}$, we can simplify Algorithm 4. In the particular case of importance weights of the criteria given be real numbers, $w_{jt} \in \mathbb{R}$ for all $j = 1, \ldots, n$ and $t = 1, \ldots, k$, the following algorithm can be considered.

Algorithm 5 *(see [23])*

Step 1. *Compute r_{ij} for $i = 1, \ldots, m$ and $j = h+1, \ldots, n$ according to (5.22)-(5.23).*

Step 2. *Compute $EV(r_{ij})$ for $j = h+1, \ldots, n$ and then $EV(r_{ij})$ for $j = 1, \ldots, h$ as*

$$EV(r_{ij}) = \frac{1}{k}\left(EV\left(r_{ij1}\right) + \ldots + EV\left(r_{ijk}\right)\right).$$

Step 3. *Compute*

$$EV(G_i) = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{1}{k}\sum_{t=1}^{k} w_{jt}\right) EV(r_{ij}). \tag{5.24}$$

Step 4. *If $EV(G_{i_1}) > EV(G_{i_2}) > \ldots > EV(G_{i_m})$ then alternatives $A_{i_1}, A_{i_2}, \ldots, A_{i_m}$ form a descending order, i.e. A_{i_1} is the best alternative and A_{i_m} is the worst one.*

Let us remark that if $(a,b,c,d)_{r,s}$ is a semi-trapezoidal fuzzy numbers of (r,s) type then

$$EV\left((a,b,c,d)_{r,s}\right) = \frac{a+rb}{2(1+r)} + \frac{d+sc}{2(1+s)}$$

and Algorithm 5 can be easily applied.

If the ratings of alternatives versus all criteria are real numbers, i.e. $r_{ijt} \in \mathbb{R}$ for all $i = 1,\ldots,m$, $j = 1,\ldots,h$, $t = 1,\ldots,k$ and $x_{ij} \in \mathbb{R}$ for all $i = 1,\ldots,m$, $j = h+1,\ldots,n$, then Algorithm 4 can be substantially simplified too. The following algorithm can be considered.

Algorithm 6 *(see [23])*

Step 1. *Compute for $i = 1,\ldots,m$, $j = 1,\ldots,h$*

$$r_{ij} = \frac{1}{k}\sum_{t=1}^{k} r_{ijt}.$$

Step 2. *Normalize x_{ij} for $i = 1,\ldots,m$, $j = h+1,\ldots,n$ by*

$$r_{ij} = \begin{cases} \frac{x_{ij}-m_j}{M_j-m_j} & \text{if} \quad j = h+1,\ldots,p \\ \frac{M_j-x_{ij}}{M_j-m_j} & \text{if} \quad j = p+1,\ldots,n, \end{cases}$$

where $m_j = \min_{i=1,\ldots,m} x_{ij}, M_j = \max_{i=1,\ldots,m} x_{ij}$.

Step 3. *Compute*

$$EV(G_i) = \frac{1}{n}\sum_{j=1}^{n} r_{ij}\left(\frac{1}{k}\sum_{t=1}^{k} EV(w_{jt})\right). \tag{5.25}$$

Step 4. *If $EV(G_{i_1}) > EV(G_{i_2}) > \ldots > EV(G_{i_m})$ then alternatives $A_{i_1}, A_{i_2}, \ldots, A_{i_m}$ form a descending order.*

5.3.3 Trapezoidal fuzzy numbers in FMDM

To simplify calculations in Algorithm 4 we can approximate the input fuzzy data by the trapezoidal, triangular or other fuzzy numbers with shapes acceptable from the fuzzy calculus point of view (see Chapter 3). Nevertheless, there exists a risk that the undertaken approximation may change the ordering generated by the expected value. Thus a good approximation $T(A)$ of a fuzzy number A must satisfy the following requirement

$$EV(A\cdot B)\leq EV(C\cdot D)\Rightarrow EV(T(A)\cdot T(B))\leq EV(T(C)\cdot T(D)),$$

for any fuzzy numbers A,B,C,D.

In this section we describe the method considered in the previous section in the case of trapezoidal fuzzy numbers. The idea of using trapezoidal fuzzy numbers is quite justified because many authors consider the linguistic variables as represented by the trapezoidal fuzzy numbers (see, e.g., [10, 82]).

Let us assume that $r_{ijt}=(e_{ijt},f_{ijt},g_{ijt},h_{ijt})$ for $i=1,\ldots,m,\ j=1,\ldots,h,\ t=1,\ldots,k$, describe the performance of alternative A_i versus subjective criterion C_j in the opinion of the decision-maker D_t. Moreover, let $x_{ij}=(a_{ij},b_{ij},c_{ij},d_{ij})$, where $i=1,\ldots,m,\ j=h+1,\ldots,p$, denote the performance of alternative A_i versus objective criterion of benefit kind C_j, while $x_{ij}=(a_{ij},b_{ij},c_{ij},d_{ij})$ for $i=1,\ldots,m,\ j=p+1,\ldots,n$, the performance of alternative A_i versus objective criterion of cost kind C_j and let $w_{jt}=(o_{jt},p_{jt},q_{jt},s_{jt})$ for $j=1,\ldots,n, t=1,\ldots,k$, be the importance weight of the criterion C_j in opinion of the decision-maker D_t.

One can show that

$$EV((t_1,t_2,t_3,t_4))=\frac{1}{4}(t_1+t_2+t_3+t_4)$$

and

$$EV\left((t_1,t_2,t_3,t_4)\cdot\left(t_1',t_2',t_3',t_4'\right)\right)=\frac{1}{12}\left(2t_1t_1'+t_1't_2+t_1t_2'+2t_2t_2'\right)+\frac{1}{12}\left(2t_3t_3'+t_3't_4+t_3t_4'+2t_4t_4'\right), \quad (5.26)$$

for every $(t_1,t_2,t_3,t_4),(t_1',t_2',t_3',t_4')\in\mathbb{F}^T(\mathbb{R})$ such that $t_1\geq 0, t_1'\geq 0$. Taking into account (5.18)-(5.21) and (5.26), the following algorithm for ordering the alternatives A_i, $i=1,\ldots,m$, can be elaborated.

Algorithm 7 *(see [23])*

Step 1. Compute

$$r_{ij}=(e_{ij},f_{ij},g_{ij},h_{ij})=\left(\sum_{t=1}^{k}\frac{e_{ijt}}{k},\sum_{t=1}^{k}\frac{f_{ijt}}{k},\sum_{t=1}^{k}\frac{g_{ijt}}{k},\sum_{t=1}^{k}\frac{h_{ijt}}{k}\right), \quad (5.27)$$

for $i = 1,\ldots,m$ and $j = 1,\ldots,h$.

Step 2. *Compute*

$$r_{ij} = (e_{ij}, f_{ij}, g_{ij}, h_{ij}) = \left(\frac{a_{ij}-a_j^*}{m_j^*}, \frac{b_{ij}-a_j^*}{m_j^*}, \frac{c_{ij}-a_j^*}{m_j^*}, \frac{d_{ij}-a_j^*}{m_j^*} \right), \tag{5.28}$$

for $i = 1,\ldots,m$ and $j = h+1,\ldots,p$ or

$$r_{ij} = (e_{ij}, f_{ij}, g_{ij}, h_{ij}) = \left(\frac{d_j^*-d_{ij}}{m_j^*}, \frac{d_j^*-c_{ij}}{m_j^*}, \frac{d_j^*-b_{ij}}{m_j^*}, \frac{d_j^*-a_{ij}}{m_j^*} \right), \tag{5.29}$$

for $i = 1,\ldots,m$ and $j = p+1,\ldots,n$, where $a_j^* = \min_{i=1,\ldots,m} a_{ij}$, $d_j^* = \max_{i=1,\ldots,m} d_{ij}$ and $m_j^* = d_j^* - a_j^*$, $j = h+1,\ldots,n$.

Step 3. *Compute*

$$w_j = (o_j, p_j, q_j, s_j) = \left(\sum_{t=1}^{k} \frac{o_{jt}}{k}, \sum_{t=1}^{k} \frac{p_{jt}}{k}, \sum_{t=1}^{k} \frac{q_{jt}}{k}, \sum_{t=1}^{k} \frac{s_{jt}}{k} \right) \tag{5.30}$$

for $j = 1,\ldots,n$.

Step 4. *Compute*

$$\begin{aligned} EV(G_i) = &\frac{1}{6n} \sum_{j=1}^{n} (e_{ij}o_j + f_{ij}p_j + g_{ij}q_j + h_{ij}s_j) \\ &+ \frac{1}{12n} \sum_{j=1}^{n} (e_{ij}p_j + f_{ij}o_j + g_{ij}s_j + h_{ij}q_j). \end{aligned} \tag{5.31}$$

Step 5. *If $EV(G_{i_1}) > EV(G_{i_2}) > \ldots > EV(G_{i_m})$ then alternatives $A_{i_1}, A_{i_2}, \ldots, A_{i_m}$ form a descending order, i.e. A_{i_1} is better than A_{i_2} and so on till A_{i_m} which is the worst alternative.*

Example 5.4. Let us assume that we accept the ambiguity with respect to ratings of the alternatives and importance weights of criteria described in Example 5.2. To avoid complications in the presentation we consider the trapezoidal fuzzy number obtained as conjunction of two triangular fuzzy numbers given by

$$(t_1, t_2, t_3) \vee (s_1, s_2, s_3) = (\min(t_1, s_1), \min(t_2, s_2), \max(t_2, s_2), \max(t_3, s_3))$$

and its extension, by associativity, to more than two triangular fuzzy numbers. Then $M \vee H$ is represented by the trapezoidal fuzzy number $(0.2, 0.5, 0.7, 1)$, $P \vee F \vee G$ is given as the trapezoidal fuzzy number $(0, 0.2, 0.8, 1)$ and so on. We resume Example 5.2 with the new data in Table 5.5 and Table 5.6. The preliminary calculations based on (5.27)-(5.30) in Step 1-Step 3 of Algorithm 7) are illustrated in Table 5.5 and Table 5.6. Then, applying (5.31) we get

$$EV(G_1) = 0.40308 < EV(G_3) = 0.46527 < EV(G_2) = 0.4966,$$

i.e. the best selection is alternative A_2 and the worst selection is A_1. □

Table 5.5 Ratings of alternatives versus criteria (see Example 5.4).

Criteria/ Alternatives	Decision-makers D_1	D_2	D_3	D_4	$r_{ij} = (e_{ij}, f_{ij}, g_{ij}, h_{ij})$
C_1/A_1	$F \vee G$	G	VG	G	(0.575, 0.825, 0.850, 1.000)
C_1/A_2	$G \vee VG$	VG	F	F	(0.500, 0.700, 0.750, 0.850)
C_1/A_3	VG	G	G	G	(0.650, 0.850, 0.850, 1.000)
C_2/A_1	G	F	VG	G	(0.575, 0.775, 0.775, 0.925)
C_2/A_2	F	G	VG	$G \vee VG$	(0.575, 0.775, 0.825, 0.925)
C_2/A_3	F	F	G	$P \vee F \vee G$	(0.300, 0.500, 0.650, 0.850)
C_3/A_1	VG	G	G	G	(0.650, 0.850, 0.850, 1.000)
C_3/A_2	G	F	VG	G	(0.575, 0.775, 0.775, 0.925)
C_3/A_3	F	$F \vee G$	G	G	(0.450, 0.650, 0.725, 0.925)
C_4/A_1	$F \vee G \vee VG$	$P \vee F$	F	F	(0.225, 0.425, 0.625, 0.775)
C_4/A_2	F	F	G	G	(0.450, 0.650, 0.650, 0.850)
C_4/A_3	G	F	G	F	(0.450, 0.650, 0.650, 0.850)
C_5/A_1	(6.0, 6.5, 7.0, 7.6)				(0, 0.150, 0.275, 0.400)
C_5/A_2	(3.6, 4.0, 4.8, 5.0)				(0.650, 0.700, 0.900, 1.000)
C_5/A_3	(4.7, 5.0, 5.5, 5.6)				(0.500, 0.525, 0.650, 0.725)

Table 5.6 The importance weights of the criteria and the aggregated weights (see Example 5.4).

Criteria	Decision-makers D_1	D_2	D_3	D_4	$w_j = (o_j, p_j, q_j, s_j)$
C_1	VH	$H \vee VH$	H	VH	(0.600, 0.850, 0.925, 1.000)
C_2	L	M	M	M	(0.150, 0.450, 0.450, 0.725)
C_3	L	L	$L \vee M \vee H$	M	(0.050, 0.350, 0.450, 0.700)
C_4	M	$H \vee VH$	VH	VH	(0.525, 0.800, 0.875, 0.950)
C_5	H	VH	VH	$M \vee H$	(0.525, 0.800, 0.850, 1.000)

At the end of this section let us remark that the fuzzy numbers are already accepted as a suitable tool in the evaluation of services quality and touristic services quality especially (see, e.g., [10, 23, 39, 40, 41, 45, 65, 66, 134, 142, 181, 186]). The method presented in Sections 5.3.2 and 5.3.3 can help us to order a set of touristic destinations according with a set of criteria. More details can be found in [23].

5.4 Approximation of intuitionistic fuzzy numbers

5.4.1 General discussion

Besides classical fuzzy sets some of their generalizations are also applied for modeling uncertain or incomplete information. One of the most popular construction considered recently are intuitionistic fuzzy sets and - in particular - intuitionistic fuzzy numbers. In this case the approximation of the membership and nonmembership function by functions with simpler and more regular shapes are also of interest (see [16, 18, 19]).

In this section we show that under some non-restrictive conditions the approximation of an intuitionistic fuzzy number is reduced to the approximation of a fuzzy number. An important benefit is that several results are obtained as immediate consequences of the methods discussed in Chapter 3. For the notation related to intuitionistic fuzzy sets we refer the reader to Section 2.1. The main ideas in the present section were developed in [26] and [28].

Theorem 5.8. *Let us consider a function* $M : \mathbb{F}_{\langle\rangle}(\mathbb{R}) \to 2^{\mathbb{F}(\mathbb{R})}$ *satisfying the following properties:*

i) For any intuitionistic fuzzy number $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle$ *we have* $M(A_{\langle\rangle}) = M(\frac{1}{2} \cdot \mu_A + \frac{1}{2} \cdot (1 - \nu_A))$.

ii) For any fuzzy number $A \in \mathbb{F}(\mathbb{R})$ *there exists a unique element* $s(A) \in M(A)$ *such that*

$$d(A, s(A)) = \min_{B \in M(A)} d(A, B).$$

Then, for any intuitionistic fuzzy number $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle$ *there exists a unique element* $S(A_{\langle\rangle}) \in M(A_{\langle\rangle})$ *such that*

$$\widetilde{d}(A_{\langle\rangle}, S(A_{\langle\rangle})) = \min_{B \in M(A_{\langle\rangle})} \widetilde{d}(A_{\langle\rangle}, B).$$

Moreover, we have $S(A_{\langle\rangle}) = s(\frac{1}{2} \cdot \mu_A + \frac{1}{2} \cdot (1 - \nu_A))$.

Proof. Let us choose arbitrary $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle \in \mathbb{F}_{\langle\rangle}(\mathbb{R})$ and $B \in M(A_{\langle\rangle})$. We have (see (1.40) and (2.3))

$$\begin{aligned}
&\widetilde{d}^2(A_{\langle\rangle}, B) \\
&= \frac{1}{2} d^2(\mu_A, B) + \frac{1}{2} d^2(1 - \nu_A, B) \\
&= \frac{1}{2} \int_0^1 ((\mu_A)_L(\alpha) - B_L(\alpha))^2 \, d\alpha + \frac{1}{2} \int_0^1 ((\mu_A)_U(\alpha) - B_U(\alpha))^2 \, d\alpha
\end{aligned}$$

$$+\frac{1}{2}\int_0^1((1-\nu_A)_L(\alpha)-B_L(\alpha))^2\,d\alpha+\frac{1}{2}\int_0^1((1-\nu_A)_U(\alpha)-B_U(\alpha))^2\,d\alpha$$
$$=d^2(\frac{1}{2}\cdot\mu_A+\frac{1}{2}\cdot(1-\nu_A),B)+\frac{1}{4}\int_0^1((\mu_A)_L(\alpha)-(1-\nu_A)_L(\alpha))^2\,d\alpha$$
$$+\frac{1}{4}\int_0^1((\mu_A)_U(\alpha)-(1-\nu_A)_U(\alpha))^2\,d\alpha.$$

Since the expression

$$\frac{1}{4}\int_0^1((\mu_A)_L(\alpha)-(1-\nu_A)_L(\alpha))^2\,d\alpha+\frac{1}{4}\int_0^1((\mu_A)_U(\alpha)-(1-\nu_A)_U(\alpha))^2\,d\alpha$$

is constant, it follows that $\widetilde{d}^2(A_{\langle\rangle},B)$ is minimal if and only if $d^2(\frac{1}{2}\cdot\mu_A+\frac{1}{2}\cdot(1-\nu_A),B)$ minimizes. Taking into account the assumptions of the theorem we easily obtain the desired conclusion. ■

Below we show how to utilize Theorem 5.8 to obtain some particular approximations of intuitionistic fuzzy numbers. Before to give the results, we introduce the following notations for an intuitionistic fuzzy number $A_{\langle\rangle}=\langle\mu_A,\nu_A\rangle$

$$m_L=\int_0^1(\mu_A)_L(\alpha)\,d\alpha, m_U=\int_0^1(\mu_A)_U(\alpha)\,d\alpha,$$
$$n_L=\int_0^1(\nu_A)_L(\alpha)\,d\alpha, n_U=\int_0^1(\nu_A)_U(\alpha)\,d\alpha,$$
$$M_L=\int_0^1\alpha\,(\mu_A)_L(\alpha)\,d\alpha, M_U=\int_0^1\alpha\,(\mu_A)_U(\alpha)\,d\alpha,$$
$$N_L=\int_0^1\alpha\,(\nu_A)_L(\alpha)\,d\alpha, N_U=\int_0^1\alpha\,(\nu_A)_U(\alpha)\,d\alpha.$$

5.4.2 Fuzzy number nearest to intuitionistic fuzzy number

Let us consider a function $M:\mathbb{F}_{\langle\rangle}(\mathbb{R})\to 2^{\mathbb{F}(\mathbb{R})}$ such that $M(A_{\langle\rangle})=\mathbb{F}(\mathbb{R})$ for all $A\in\mathbb{F}_{\langle\rangle}(\mathbb{R})$. It is immediate that the assumptions in Theorem 5.8 are satisfied and since in this case $s(A)=A$ for all $A\in\mathbb{F}(\mathbb{R})$ we obtain the following theorem.

Theorem 5.9. *If $A_{\langle\rangle}=\langle\mu_A,\nu_A\rangle$ is an intuitionistic fuzzy number then*

$$S(A_{\langle\rangle})=\frac{1}{2}\cdot\mu_A+\frac{1}{2}\cdot(1-\nu_A)$$

is the nearest fuzzy number to $A_{\langle\rangle}$ with respect to the distance $\widetilde{d}$ and $S(A_{\langle\rangle})$ is unique with this property.

We easily conclude that if $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle$ is a trapezoidal intuitionistic fuzzy number, i.e. $\mu_A = (t_1, t_2, t_3, t_4)$ and $1 - \nu_A = (s_1, s_2, s_3, s_4)$, then

$$S\left(A_{\langle\rangle}\right) = \left(\frac{t_1 + s_1}{2}, \frac{t_2 + s_2}{2}, \frac{t_3 + s_3}{2}, \frac{t_4 + s_4}{2}\right)$$

is the nearest trapezoidal fuzzy number to $A_{\langle\rangle}$.

Now let us consider a trapezoidal approximation of an intuitionistic fuzzy number. Suppose $M : \mathbb{F}_{\langle\rangle}(\mathbb{R}) \to 2^{\mathbb{F}(\mathbb{R})}$ is a function such that $M(A_{\langle\rangle}) = \mathbb{F}^T(\mathbb{R})$, for all $A_{\langle\rangle} \in \mathbb{F}_{\langle\rangle}(\mathbb{R})$. For $A \in \mathbb{F}(\mathbb{R})$ let us consider $s(A) = t(A)$, where t is the trapezoidal approximation of A with respect to the distance d. By Theorem 5.8 we get the following result.

Theorem 5.10. *If $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle$ is an intuitionistic fuzzy number then*

$$S(A_{\langle\rangle}) = t\left(\frac{1}{2} \cdot \mu_A + \frac{1}{2} \cdot (1 - \nu_A)\right)$$

is the nearest trapezoidal fuzzy number to $A_{\langle\rangle}$ with respect to the distance $\widetilde{d}$ and it is unique with this property.

The nearest trapezoidal fuzzy number to an intuitionistic fuzzy number was calculated in [26] by applying the Karush-Kuhn-Tucker theorem. It can be obtained directly by Theorem 5.10 and taking into account Theorem 4.4 in [195]. We get the following result

Theorem 5.11. *Let $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle$ be an intuitionistic fuzzy number and*

$$T\left(A_{\langle\rangle}\right) = \left(t_1\left(A_{\langle\rangle}\right), t_2\left(A_{\langle\rangle}\right), t_3\left(A_{\langle\rangle}\right), t_4\left(A_{\langle\rangle}\right)\right) = (t_1, t_2, t_3, t_4)$$

the nearest (with respect to the metric $\widetilde{d}$) trapezoidal fuzzy number to $A_{\langle\rangle}$.

(i) *If*

$$-m_L + m_U + 2n_L - 2n_U + 3M_L - 3N_L - 3M_U + 3N_U \leq 0$$

then

$$t_1 = 2m_L - n_L - 3M_L + 3N_L, \tag{5.32}$$
$$t_2 = -m_L + 2n_L + 3M_L - 3N_L, \tag{5.33}$$
$$t_3 = -m_U + 2n_U + 3M_U - 3N_U, \tag{5.34}$$
$$t_4 = 2m_U - n_U - 3M_U + 3N_U. \tag{5.35}$$

(ii) *If*

$$-m_L - 3m_U + 2n_L + 2n_U + 3M_L + 5M_U - 3N_L - 5N_U > 0$$

then

$$t_1 = \frac{8}{5}m_L - \frac{1}{5}m_U - \frac{9}{5}M_L - \frac{1}{5}n_L - \frac{1}{5}n_U + \frac{9}{5}N_L, \tag{5.36}$$

$$t_2 = t_3 = t_4 = -\frac{1}{5}m_L + \frac{2}{5}m_U + \frac{3}{5}M_L + \frac{2}{5}n_L + \frac{2}{5}n_U - \frac{3}{5}N_L. \tag{5.37}$$

(iii) *If*

$$-3m_L - m_U + 2n_L + 2n_U + 5M_L + 3M_U - 5N_L - 3N_U < 0$$

then

$$t_1 = t_2 = t_3 = \frac{2}{5}m_L - \frac{1}{5}m_U + \frac{3}{5}M_U + \frac{2}{5}n_L + \frac{2}{5}n_U - \frac{3}{5}N_U, \tag{5.38}$$

$$t_4 = -\frac{1}{5}m_L + \frac{8}{5}m_U - \frac{9}{5}M_U - \frac{1}{5}n_L - \frac{1}{5}n_U + \frac{9}{5}N_U. \tag{5.39}$$

(iv) *If*

$$-m_L + m_U + 2n_L - 2n_U + 3M_L - 3N_L - 3M_U + 3N_U > 0,$$
$$-m_L - 3m_U + 2n_L + 2n_U + 3M_L + 5M_U - 3N_L - 5N_U \leq 0,$$

and

$$-3m_L - m_U + 2n_L + 2n_U + 5M_L + 3M_U - 5N_L - 3N_U \geq 0$$

then

$$t_1 = \frac{7}{4}m_L + \frac{1}{4}m_U - \frac{9}{4}M_L - \frac{3}{4}M_U - \frac{1}{2}n_L - \frac{1}{2}n_U + \frac{9}{4}N_L + \frac{3}{4}N_U, \tag{5.40}$$

$$t_2 = t_3 = -\frac{1}{2}m_L - \frac{1}{2}m_U + n_L + n_U + \frac{3}{2}M_L + \frac{3}{2}M_U - \frac{3}{2}N_L - \frac{3}{2}N_U, \tag{5.41}$$

$$t_4 = \frac{1}{4}m_L + \frac{7}{4}m_U - \frac{3}{4}M_L - \frac{9}{4}M_U - \frac{1}{2}n_L - \frac{1}{2}n_U + \frac{3}{4}N_L + \frac{9}{4}N_U. \tag{5.42}$$

Another approach is connected with the trapezoidal approximation of an intuitionistic fuzzy number preserving the expected interval (see [18]). Let us consider the function $M : \mathbb{F}_{\langle\rangle}(\mathbb{R}) \to 2^{\mathbb{F}(\mathbb{R})}$ such that $M(A_{\langle\rangle}) = \{A \in \mathbb{F}(\mathbb{R}) : EI(A_{\langle\rangle}) = EI(A)\}$. The equality $M(A_{\langle\rangle}) = M(\frac{1}{2} \cdot \mu_A + \frac{1}{2} \cdot (1 - \nu_A))$ is immediate due to (2.4). Given any $A \in \mathbb{F}(\mathbb{R})$ let us denote by $t_{ei}(A)$ the nearest trapezoidal fuzzy number to A (with respect to the metric d) preserving the expected interval of A. As a conclusion of Theorem 5.8 we get the following theorem.

Theorem 5.12. *If $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle$ is an intuitionistic fuzzy number then*

$$S(A_{\langle\rangle}) = t_{ei}(\frac{1}{2} \cdot \mu_A + \frac{1}{2} \cdot (1 - \nu_A))$$

is the nearest trapezoidal fuzzy number to $A_{\langle\rangle}$ (with respect to the distance $\widetilde{d}$), preserving the expected interval of $A_{\langle\rangle}$ and it is unique with this property.

The approach in [18] is based on Karush-Kuhn-Tucker theorem (see Section 3.5.1) and it is complicated. We obtain the same result from Theorem 7 in [17] as follows

Theorem 5.13. *Let $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle$ be an intuitionistic fuzzy number and*

$$T_{ei}\left(A_{\langle\rangle}\right) = \left(t_1\left(A_{\langle\rangle}\right), t_2\left(A_{\langle\rangle}\right), t_3\left(A_{\langle\rangle}\right), t_4\left(A_{\langle\rangle}\right)\right) = (t_1, t_2, t_3, t_4)$$

the nearest (with respect to the metric $\widetilde{d}$) trapezoidal fuzzy number to $A_{\langle\rangle}$ which preserves its expected interval.

(i) *If*

$$-m_L + m_U + 2n_L - 2n_U + 3M_L - 3N_L - 3M_U + 3N_U \le 0$$

then

$$t_1 = 2m_L - n_L - 3M_L + 3N_L, \tag{5.43}$$
$$t_2 = -m_L + 2n_L + 3M_L - 3N_L, \tag{5.44}$$
$$t_3 = -m_U + 2n_U + 3M_U - 3N_U, \tag{5.45}$$
$$t_4 = 2m_U - n_U - 3M_U + 3N_U. \tag{5.46}$$

(ii) *If*

$$-m_L - 2m_U + 2n_L + n_U + 3M_L + 3M_U - 3N_L - 3N_U > 0$$

then

$$t_1 = m_L - \frac{1}{2}m_U + n_L - \frac{1}{2}n_U, \tag{5.47}$$
$$t_2 = t_3 = t_4 = \frac{1}{2}m_U + \frac{1}{2}n_U. \tag{5.48}$$

(iii) *If*

$$-2m_L - m_U + n_L + 2n_U + 3M_L + 3M_U - 3N_L - 3N_U < 0$$

then

$$t_1 = t_2 = t_3 = \frac{1}{2}m_L + \frac{1}{2}n_L, \tag{5.49}$$
$$t_4 = -\frac{1}{2}m_L + m_U - \frac{1}{2}n_L + n_U. \tag{5.50}$$

(iv) If

$$-m_L + m_U + 2n_L - 2n_U + 3M_L - 3N_L - 3M_U + 3N_U > 0,$$
$$-m_L - 2m_U + 2n_L + n_U + 3M_L + 3M_U - 3N_L - 3N_U \le 0$$

and

$$-2m_L - m_U + n_L + 2n_U + 3M_L + 3M_U - 3N_L - 3N_U \ge 0$$

then

$$t_1 = \frac{3}{2}m_L + \frac{1}{2}m_U - \frac{3}{2}M_L - \frac{3}{2}M_U - n_U + \frac{3}{2}N_L + \frac{3}{2}N_U, \tag{5.51}$$

$$t_2 = t_3 = -\frac{1}{2}m_L - \frac{1}{2}m_U + n_L + n_U \tag{5.52}$$
$$+ \frac{3}{2}M_L + \frac{3}{2}M_U - \frac{3}{2}N_L - \frac{3}{2}N_U,$$

$$t_4 = \frac{1}{2}m_L + \frac{3}{2}m_U - \frac{3}{2}M_L - \frac{3}{2}M_U - n_L + \frac{3}{2}N_L + \frac{3}{2}N_U. \tag{5.53}$$

We may also consider a trapezoidal approximation of an intuitionistic fuzzy number preserving value and ambiguity. It is immediate that (see (2.7) and (2.8))

$$Amb(A_{\langle\rangle}) = Amb(\frac{1}{2} \cdot \mu_A + \frac{1}{2} \cdot (1 - \nu_A)) \tag{5.54}$$

and

$$Val(A_{\langle\rangle}) = Val(\frac{1}{2} \cdot \mu_A + \frac{1}{2} \cdot (1 - \nu_A)), \tag{5.55}$$

for every $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle \in \mathbb{F}_{\langle\rangle}(\mathbb{R})$. Now let us consider a function $M : \mathbb{F}_{\langle\rangle}(\mathbb{R}) \to 2^{\mathbb{F}(\mathbb{R})}$ such that $M(A_{\langle\rangle}) = \{A \in \mathbb{F}(\mathbb{R}) : Val(A_{\langle\rangle}) = Val(A) \text{ and } Amb(A_{\langle\rangle}) = Amb(A)\}$. We have $M(A_{\langle\rangle}) = M(\frac{1}{2} \cdot \mu_A + \frac{1}{2} \cdot (1 - \nu_A))$. If we denote by $t_{a,v}(A)$ the nearest trapezoidal approximation of $A \in \mathbb{F}(\mathbb{R})$ (with respect to d) preserving the value and ambiguity of A then by Theorem 5.8 we may prove the following theorem.

Theorem 5.14. *If $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle$ is an intuitionistic fuzzy number then*

$$S(A_{\langle\rangle}) = t_{a,v}(\frac{1}{2} \cdot \mu_A + \frac{1}{2} \cdot (1 - \nu_A))$$

is the nearest trapezoidal fuzzy number to $A_{\langle\rangle}$ (with respect to the distance $\widetilde{d}$) preserving the value and ambiguity of $A_{\langle\rangle}$ and it is unique with this property.

The nearest trapezoidal fuzzy number to a given fuzzy number (with respect to the distance d) preserving the value and ambiguity was discussed firstly in [25]. Taking into account Theorem 5.14 we immediately obtain

Theorem 5.15. *Let $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle$ be an intuitionistic fuzzy number and let*

$$T_{a,v}\left(A_{\langle\rangle}\right) = \left(t_1\left(A_{\langle\rangle}\right), t_2\left(A_{\langle\rangle}\right), t_3\left(A_{\langle\rangle}\right), t_4\left(A_{\langle\rangle}\right)\right) = (t_1, t_2, t_3, t_4)$$

denote the trapezoidal fuzzy number nearest to $A_{\langle\rangle}$ *(with respect to the metric* $\widetilde{d}$*) which preserves its value and ambiguity.*

(i) If

$$-m_L + m_U + 2n_L - 2n_U + 3M_L - 3N_L - 3M_U + 3N_U \leq 0$$

then

$$\begin{aligned} t_1 &= 2m_L - n_L - 3M_L + 3N_L, \\ t_2 &= -m_L + 2n_L + 3M_L - 3N_L, \\ t_3 &= -m_U + 2n_U + 3M_U - 3N_U, \\ t_4 &= 2m_U - n_U - 3M_U + 3N_U. \end{aligned}$$

(ii) If

$$-m_L - m_U + 2n_L + 3M_L + M_U - 3N_L - N_U > 0$$

then

$$\begin{aligned} t_1 &= 3n_L - 2n_U + 3M_L - 2M_U - 3N_L + 2N_U, \\ t_2 &= t_3 = t_4 = n_U + M_U - N_U. \end{aligned}$$

(iii) If

$$-m_L - m_U + 2n_U + M_L + 3M_U - N_L - 3N_U < 0$$

then

$$\begin{aligned} t_1 &= t_2 = t_3 = n_L + M_L - N_L, \\ t_4 &= -2n_L + 3n_U - 2M_L + 3M_U + 2N_L - 3N_U. \end{aligned}$$

(iv) If

$$\begin{aligned} -m_L + m_U + 2n_L - 2n_U + 3M_L - 3M_U - 3N_L + 3N_U &> 0, \\ -m_L - m_U + 2n_L + 3M_L + M_U - 3N_L - N_U &\leq 0 \end{aligned}$$

and

$$-m_L - m_U + 2n_U + M_L + 3M_U - N_L - 3N_U \geq 0$$

then

$$\begin{aligned} t_1 &= m_L + m_U + n_L - 2n_U - 3M_U + 3N_U, \\ t_2 &= t_3 = -\frac{1}{2}m_L - \frac{1}{2}m_U + n_L + n_U \\ &\quad + \frac{3}{2}M_L + \frac{3}{2}M_U - \frac{3}{2}N_L - \frac{3}{2}N_U, \\ t_4 &= m_L + m_U - 2n_L + n_U - 3M_L + 3N_L. \end{aligned}$$

Sometimes one may be interested in trapezoidal approximation of an intuitionistic fuzzy number preserving core. It is immediate that (see (2.11))

$$\operatorname{core}\left(A_{\langle\rangle}\right) = \operatorname{core}(\frac{1}{2} \cdot \mu_A + \frac{1}{2} \cdot (1 - \nu_A))$$

for every $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle \in \mathbb{F}_{\langle\rangle}(\mathbb{R})$. Let us consider a function $M : \mathbb{F}_{\langle\rangle}(\mathbb{R}) \to 2^{\mathbb{F}(\mathbb{R})}$ such that $M(A_{\langle\rangle}) = \{A \in \mathbb{F}(\mathbb{R}) : \operatorname{core}(A_{\langle\rangle}) = \operatorname{core}(A)\}$. Thus we have $M(A_{\langle\rangle}) = M(\frac{1}{2} \cdot \mu_A + \frac{1}{2} \cdot (1 - \nu_A))$. If we denote by $t_c(A)$ the nearest trapezoidal fuzzy number to $A \in \mathbb{F}(\mathbb{R})$ (with respect to the metric d) preserving the core of A then by Theorem 5.8 we obtain the following theorem.

Theorem 5.16. *If $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle$ is an intuitionistic fuzzy number then*

$$S(A_{\langle\rangle}) = t_c(\frac{1}{2} \cdot \mu_A + \frac{1}{2} \cdot (1 - \nu_A))$$

is the nearest trapezoidal fuzzy number to $A_{\langle\rangle}$ (with respect to the distance $\widetilde{d}$) preserving the core of $A_{\langle\rangle}$ and it is unique with this property.

The nearest trapezoidal fuzzy number to a given fuzzy number preserving the core was computed in [5]. Taking into account Theorem 5.16 we immediately obtain the following approximation.

Theorem 5.17. *The nearest trapezoidal fuzzy number to an intuitionistic fuzzy number $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle$ (with respect to the metric $\widetilde{d}$) which preserves the core of $A_{\langle\rangle}$, denoted by $T_c(A_{\langle\rangle}) = \left(t_1\left(A_{\langle\rangle}\right), t_2\left(A_{\langle\rangle}\right), t_3\left(A_{\langle\rangle}\right), t_4\left(A_{\langle\rangle}\right)\right) = (t_1, t_2, t_3, t_4)$, is given by*

$$\begin{aligned}
t_1 &= \frac{3}{2}\int_0^1 (\mu_A)_L(\alpha)\,d\alpha - \frac{3}{2}\int_0^1 \alpha\,(\mu_A)_L(\alpha)\,d\alpha \\
&\quad + \frac{3}{2}\int_0^1 \alpha\,(\nu_A)_L(\alpha)\,d\alpha - \frac{1}{4}\left((\mu_A)_L(1) + (\nu_A)_L(0)\right), \\
t_2 &= \frac{1}{2}\left((\mu_A)_L(1) + (\nu_A)_L(0)\right), \\
t_3 &= \frac{1}{2}\left((\mu_A)_U(1) + (\nu_A)_U(0)\right), \\
t_4 &= -\frac{3}{2}\int_0^1 \alpha\,(\mu_A)_U(\alpha)\,d\alpha + \frac{3}{2}\int_0^1 \alpha\,(\nu_A)_U(\alpha)\,d\alpha \\
&\quad + \frac{3}{2}\int_0^1 (\mu_A)_U(\alpha)\,d\alpha - \frac{1}{4}\left((\mu_A)_U(1) + (\nu_A)_U(0)\right).
\end{aligned}$$

5.4.3 Transfer of properties

With respect to the properties of the mappings $s : \mathbb{F}(\mathbb{R}) \to \mathbb{F}(\mathbb{R})$ and $S : \mathbb{F}_{\langle\rangle}(\mathbb{R}) \to \mathbb{F}(\mathbb{R})$ in Theorem 5.8 we can formulate the following result.

Theorem 5.18.

(i) If s is additive then S is also additive.
(ii) If s is invariant to translations then S is also invariant to translations.
(iii) If s is scale invariant then S is also scale invariant.
(iv) If s has the Lipschitz constant c then S has the same Lipschitz constant c.
(v) If s is continuous then S is also continuous.

Proof. (i) Let $A, B \in \mathbb{F}_{\langle\rangle}(\mathbb{R})$, $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle$ and $B_{\langle\rangle} = \langle \mu_B, \nu_B \rangle$. According to the definition of addition (see Section 2.1) we have $A_{\langle\rangle} + B_{\langle\rangle} = (A+B)_{\langle\rangle} = \langle \mu_{A+B}, \nu_{A+B} \rangle$, where $\mu_{A+B} = \mu_A + \mu_B, 1 - \nu_{A+B} = (1-\nu_A) + (1-\nu_B)$. Taking into account Theorem 5.8 and the assumption we get

$$\begin{aligned} S(A_{\langle\rangle} + B_{\langle\rangle}) &= s\left(\frac{1}{2}\cdot(\mu_A+\mu_B) + \frac{1}{2}\cdot((1-\nu_A)+(1-\nu_B))\right) \\ &= s\left(\frac{1}{2}\cdot\mu_A + \frac{1}{2}\cdot(1-\nu_A)\right) + s\left(\frac{1}{2}\cdot\mu_B + \frac{1}{2}\cdot(1-\nu_B)\right) \\ &= S(A_{\langle\rangle}) + S(B_{\langle\rangle}). \end{aligned}$$

(ii) It is immediate from (i).

(iii) Let $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle \in \mathbb{F}_{\langle\rangle}(\mathbb{R})$ and $\lambda \in \mathbb{R}$. The definition of the scalar multiplication of intuitionistic fuzzy numbers (see Section 2.1) and Theorem 5.8 imply

$$\begin{aligned} S(\lambda \cdot A_{\langle\rangle}) &= S(\langle \lambda\cdot\mu_A, 1-\lambda\cdot(1-\nu_A)\rangle) \\ &= s\left(\frac{1}{2}\lambda\cdot\mu_A + \frac{1}{2}\lambda\cdot(1-\nu_A)\right) \\ &= s\left(\lambda\cdot(\frac{1}{2}\cdot\mu_A + \frac{1}{2}\cdot(1-\nu_A))\right) \\ &= \lambda\cdot s\left(\frac{1}{2}\cdot\mu_A + \frac{1}{2}\cdot(1-\nu_A)\right) \\ &= \lambda\cdot S(A_{\langle\rangle}). \end{aligned}$$

(iv) Let $A_{\langle\rangle}, B_{\langle\rangle} \in \mathbb{F}_{\langle\rangle}(\mathbb{R})$, $A_{\langle\rangle} = \langle \mu_A, \nu_A \rangle$ and $B_{\langle\rangle} = \langle \mu_B, \nu_B \rangle$. By Theorem 5.8 we get

$$S(A_{\langle\rangle}) = s\left(\frac{1}{2}\cdot\mu_A + \frac{1}{2}\cdot(1-\nu_A)\right)$$

and

$$S(B_{\langle\rangle}) = s\left(\frac{1}{2}\cdot\mu_B + \frac{1}{2}\cdot(1-\nu_B)\right).$$

If s is Lipschitz with the constant c then

$$d\left(s\left(\frac{1}{2}\cdot\mu_A+\frac{1}{2}\cdot(1-\nu_A)\right),s\left(\frac{1}{2}\cdot\mu_B+\frac{1}{2}\cdot(1-\nu_B)\right)\right)$$
$$\leq cd\left(\frac{1}{2}\cdot\mu_A+\frac{1}{2}\cdot(1-\nu_A),\frac{1}{2}\cdot\mu_B+\frac{1}{2}\cdot(1-\nu_B)\right).$$

On the other hand

$$\begin{aligned}
&d^2\left(\frac{1}{2}\cdot\mu_A+\frac{1}{2}\cdot(1-\nu_A),\frac{1}{2}\cdot\mu_B+\frac{1}{2}\cdot(1-\nu_B)\right)\\
&=\int_0^1\left(\left(\frac{1}{2}\cdot\mu_A+\frac{1}{2}\cdot(1-\nu_A)\right)_L(\alpha)-\left(\frac{1}{2}\cdot\mu_B+\frac{1}{2}\cdot(1-\nu_B)\right)_L(\alpha)\right)^2d\alpha\\
&+\int_0^1\left(\left(\frac{1}{2}\cdot\mu_A+\frac{1}{2}\cdot(1-\nu_A)\right)_U(\alpha)-\left(\frac{1}{2}\cdot\mu_B+\frac{1}{2}\cdot(1-\nu_B)\right)_U(\alpha)\right)^2d\alpha\\
&=\int_0^1\left(\frac{1}{2}(\mu_A)_L(\alpha)+\frac{1}{2}(1-\nu_A)_L(\alpha)-\frac{1}{2}(\mu_B)_L(\alpha)-\frac{1}{2}(1-\nu_B)_L(\alpha)\right)^2d\alpha\\
&+\int_0^1\left(\frac{1}{2}(\mu_A)_U(\alpha)+\frac{1}{2}(1-\nu_A)_U(\alpha)-\frac{1}{2}(\mu_B)_U(\alpha)-\frac{1}{2}(1-\nu_B)_U(\alpha)\right)^2d\alpha\\
&=\frac{1}{4}\int_0^1((\mu_A)_L(\alpha)-(\mu_B)_L(\alpha)+(1-\nu_A)_L(\alpha)-(1-\nu_B)_L(\alpha))^2d\alpha\\
&+\frac{1}{4}\int_0^1((\mu_A)_U(\alpha)-(\mu_B)_U(\alpha)+(1-\nu_A)_U(\alpha)-(1-\nu_B)_U(\alpha))^2d\alpha\\
&\leq\frac{1}{2}\int_0^1((\mu_A)_L(\alpha)-(\mu_B)_L(\alpha))^2d\alpha+\frac{1}{2}\int_0^1((1-\nu_A)_L(\alpha)-(1-\nu_B)_L(\alpha))^2d\alpha\\
&+\frac{1}{2}\int_0^1(((\mu_A)_U(\alpha)-(\mu_B)_U(\alpha))^2d\alpha+\frac{1}{2}\int_0^1((1-\nu_A)_U(\alpha)-(1-\nu_B)_U(\alpha))^2d\alpha\\
&=\frac{1}{2}d^2(\mu_A,\mu_B)+\frac{1}{2}d^2(1-\nu_A,1-\nu_B)\\
&=\widetilde{d}^2(A_{\langle\rangle},B_{\langle\rangle}).
\end{aligned}$$

Hence we get $\widetilde{d}\left(S(A_{\langle\rangle}),S(B_{\langle\rangle})\right)\leq c\widetilde{d}(A_{\langle\rangle},B_{\langle\rangle})$.

(v) We have $S=s\circ f$, where $f:\mathbb{F}_{\langle\rangle}(\mathbb{R})\to\mathbb{F}(\mathbb{R})$ is defined by $f(A_{\langle\rangle})=f(\langle\mu_A,\nu_A\rangle)=\frac{1}{2}\cdot\mu_A+\frac{1}{2}\cdot(1-\nu_A)$. Since s and f are continuous we obtain the continuity of S. ■

Lists of criteria which a crisp approximation (or defuzzification) and a trapezoidal approximation operator on fuzzy numbers should possess or just possesses were proposed in [145] and [118] (see also Section 3.2). They include additivity, invariance to translations, scale invariance and continuity. The following results are immediate by Theorem 5.18.

Corollary 5.6. *(i) The operator $T:\mathbb{F}_{\langle\rangle}(\mathbb{R})\to\mathbb{F}^T(\mathbb{R})$ in Theorem 5.11 is invariant to translations, scale invariant and continuous.*

(ii) The operator $T_{ei}:\mathbb{F}_{\langle\rangle}(\mathbb{R})\to\mathbb{F}^T(\mathbb{R})$ *in Theorem 5.13 is invariant to translations, scale invariant and continuous.*

(iii) The operator $T_{a,v}:\mathbb{F}_{\langle\rangle}(\mathbb{R})\to\mathbb{F}^T(\mathbb{R})$ *in Theorem 5.14 is invariant to translations, scale invariant and continuous.*

(iv) The operator $T_c:\mathbb{F}_{\langle\rangle}(\mathbb{R})\to\mathbb{F}^T(\mathbb{R})$ *in Theorem 5.17 is invariant to translations and scale invariant.*

Proof. (i) It is immediate from Theorem 5.18 taking into account that the operator $t:\mathbb{F}(\mathbb{R})\to\mathbb{F}^T(\mathbb{R})$, where $t(A)$ is the nearest (with respect to the distance d) trapezoidal fuzzy number to fuzzy number A is invariant to translations, scale invariant and continuous (see [195]).

(ii) It is immediate from Theorem 5.18 taking into account that the operator $t_e:\mathbb{F}(\mathbb{R})\to\mathbb{F}^T(\mathbb{R})$, where $t_e(A)$ is the nearest (with respect to the distance d) trapezoidal fuzzy number to fuzzy number A, preserving the expected interval of A, is invariant to translations, scale invariant and continuous (see [17], [27]).

(iii) It is immediate from Theorem 5.18 taking into account that the operator $t_{a,v}:\mathbb{F}(\mathbb{R})\to\mathbb{F}^T(\mathbb{R})$, where $t_{a,v}(A)$ is the nearest (with respect to the distance d) trapezoidal fuzzy number to fuzzy number A, preserving the value and ambiguity of A, is invariant to translations, scale invariant and continuous (see [25]).

(iv) It is immediate from Theorem 5.18 taking into account that the operator $t_c:\mathbb{F}(\mathbb{R})\to\mathbb{F}^T(\mathbb{R})$, where $t_c(A)$ is the nearest (with respect to the distance d) trapezoidal fuzzy number to fuzzy number A preserving the core of A, is invariant to translations and scale invariant (see [5]). ■

Problems

5.1. Let A_1, A_2 be fuzzy numbers given by their α-cuts, $(A_1)_\alpha=\left[\sqrt{\alpha},3-\sqrt{\alpha}\right]$, $(A_2)_\alpha=\left[2+\sqrt{\alpha},5-2\sqrt{\alpha}\right]$, $\alpha\in[0,1]$. Find a trapezoidal fuzzy number nearest to A and B. Find also a trapezoidal fuzzy number nearest to A and B which preserves the expected interval of the set $\{A,B\}$ and verify their equality.

5.2. Find a trapezoidal fuzzy number nearest to trapezoidal fuzzy numbers $A_1=(-2,0,1,3)$, $A_2=(0,2,4,6)$ and $A_3=(1,3,4,5)$ and the trapezoidal fuzzy number nearest to A_1,A_2 and A_3 which preserves the expected interval.

5.3. Let d be a metric on $\mathbb{F}(\mathbb{R})$ which satisfies the assumptions of Theorem 5.3. Consider a sample of fuzzy numbers $\mathbb{A}=(A_1,\ldots,A_n)$. Let us denote by $M(\mathbb{A},T)$ a set of all trapezoidal medians of the sample $\mathbb{A}$ with respect to d. Prove that $M(\mathbb{A},T)$ is a compact subset of $\mathbb{F}(\mathbb{R})$ in the topology generated by d.

5.4. Consider the L^1-type metric d_1 on $\mathbb{F}(\mathbb{R})$, where

$$d_1(A,B)=\int_0^1|A_L(\alpha)-B_L(\alpha)|\,d\alpha+\int_0^1|A_U(\alpha)-B_U(\alpha)|\,d\alpha.$$

Find an example where the crisp fuzzy median of a sample is not unique with respect to d_1.

5.5. Prove that the best alternative is A_2 and the worst alternative is A_1 by considering the ratings of alternatives A_1, A_2, A_3 versus criteria C_1, C_2, C_5 given in Table 5.3 and the importance weights of criteria C_1, C_2, C_5 are $0.9, 0.4, 0.7$ with respect to decision maker D_1 and $0.8, 0.5, 0.9$ with respect to decision maker D_2.

5.6. We consider the ratings of alternatives versus criteria as in Table 5.7 and the importance weights of the criteria in Table 5.4. Prove that the best alternative is A_2 and the worst alternative is A_1.

Table 5.7 Ratings of alternatives versus criteria (see Problem 5.6).

	Decision-makers		
Criteria/Alternatives	D_1	D_2	r_{ij}
C_1/A_1	0.7	0.8	0.75
C_1/A_2	0.8	0.9	0.85
C_1/A_3	0.9	0.8	0.85
C_2/A_1	0.8	0.5	0.65
C_2/A_2	0.5	0.8	0.65
C_2/A_3	0.5	0.5	0.5
C_5/A_1	7.0		0
C_5/A_2	4.0		1
C_5/A_3	5.0		0.667

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